Quantum signal processing without angle finding

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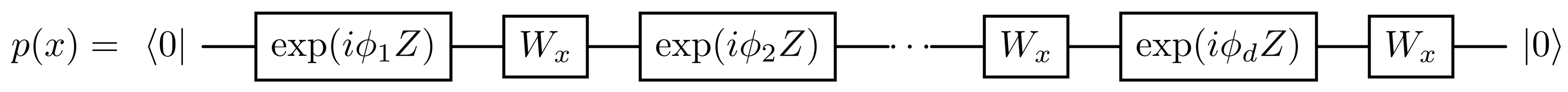
Abstract

Quantum signal processing (QSP) refers to the task of implementing a given function f on a quantum signal, which typically corresponds to the eigenvalues of an input Hermitian matrix H. QSP is employed in many modern fault-tolerant quantum algorithms, including those for Hamiltonian simulation, matrix inversion, solving differential equations and optimization. The quantum circuits proposed in the literature for implementing QSP for a polynomial function have optimal size. However, given a general function, computing the description of a circuit that implements a polynomial approximation of the function with optimal error scaling requires calculation of certain rotation angles on a classical computer, which limits the overall complexity of QSP. In this work, we use ideas from the theory of interpolating polynomials to construct a simple circuit for implementing QSP without angle finding. This circuit enables implementation of QSP for any continuous black-box function f with nearly optimal complexity, including the classical operations required for computing a description of the circuit.

Background: Quantum signal processing

- Given a degree-d real polynomial $p(x)$ with $\|p(x)\|_\infty := \max_{[-1,1]} |p(x)| \leq 1/2$, a set of angles $\Phi = (\phi_1, \phi_2, \ldots, \phi_d)$ can be calculated such that [1, 2]



Here $W_x = \begin{bmatrix} x & i\sqrt{1-x^2} \\ i\sqrt{1-x^2} & x \end{bmatrix}$ is the parametrized quantum walk unitary.

- Deterministic angle-finding algorithms for a given polynomial have complexity scaling as $O(d^2)$ or worse [2].
- For implementing a degree-d approximation of $f : [-1,1] \to \mathbb{C}$, we minimize with respect to a cost function [3, 4], e.g.

$$\min_{\Phi} \quad \|f - p_\Phi\|_\infty \quad \text{s.t.} \quad \Phi \in [-\pi, \pi)^d.$$

The complexity of this optimization for a general f is unknown.

- Suppose an $(n+a)$-qubit block encoding of an n-qubit Hamiltonian H is given. To obtain a circuit for the block encoding of $p(H)$, we replace each W_x by $(2\Pi - \mathbb{1})U_H$, where $\Pi = |0^a\rangle\langle 0^a| \otimes \mathbb{1}$.

New polynomial approximation

The key tool in our work is a new polynomial approximation that is easy to compute and has the same error scaling as the best polynomial approximation.

Lemma: For $f : [-1,1] \to \mathbb{C}$ and $d \in \mathbb{Z}^+$, define $f_d : [-1,1] \to \mathbb{C}$ as

$$f_d(x) = \frac{1}{8d^2} \sum_{j'=d}^{3d-1} \sum_{j,k=0}^{4d-1} f(x_k) \exp\left(\frac{i\pi k(j'-j)}{2d}\right) T_{|j-j'|}(x), \quad x_k = \cos\left(\frac{\pi k}{2d}\right),$$

where $\{T_j\}$ denote Cheyshev polynomials of the first kind. Then

$$\|f - f_d\|_\infty \leq (1+\sqrt{2})E_d(f),$$

where $E_d(f) := \|f - f_{d,*}\|_\infty$ and $f_{d,*}$ is the best polynomial approximation of degree d for f in $\|\cdot\|_\infty$ norm.

Remark: f_d is a polynomial of degree at most $3d-1$.
Remark: If f is a degree-d polynomial, then $f_d = f$.

Quantum signal processing for black-box functions without angle-finding

Theorem: Given an oracle O_f for a function $f : [-1,1] \to \mathbb{C}$ with $\|f\|_\infty \leq 1$ and an $(n+a)$-qubit block encoding U_H for an n-qubit Hamiltonian H, a block encoding $U_{f(H)}$ of $f(H)/\sqrt{2}$ to accuracy $(1+\sqrt{2})E_d(f) + \epsilon$ can be constructed using $\mathbf{O}(d)$ queries to U_H, two queries to O_f, and $\mathbf{O}(a, \text{polylog}(d, 1/\epsilon))$ additional two-qubit gates. Moreover, a circuit description for implementing $U_{f(H)}$ can be computed in polylog $(d, 1/\epsilon)$ time on a classical computer.

Proof: We first construct a circuit to implement $f_d(x)$:

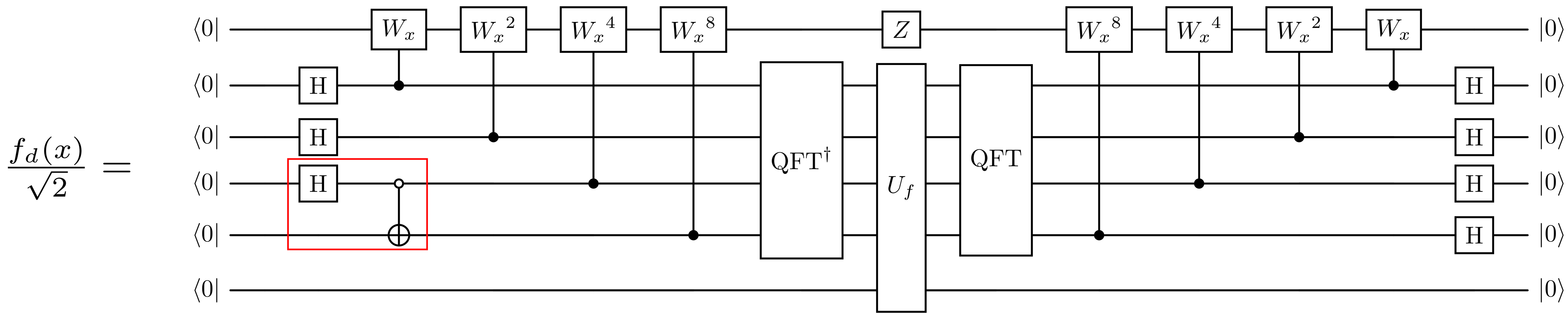


Figure: Circuit diagram for black-box QSP acting on 1 signal qubit (the top wire) and $3 + \lceil \log_2(d) \rceil$ ancilla qubits (here $d = 4$).

Here U_f has action $U_f |j\rangle|0\rangle = |j\rangle \left(f(x_j)|0\rangle + \sqrt{1 - |f(x_j)|^2}|1\rangle \right)$ and can be implemented using two queries to O_f. A circuit for $U_{f(H)}$ is obtained by $W_x \mapsto (2\Pi - \mathbb{1})U_H$.
Remark: The circuit for implementing $f_d(x)$ is remarkably similar to that for function implementation based on quantum phase estimation, but it is *exact* for degree-d polynomials.

Summary and discussion

- Constructed a polynomial approximation that is easy to implement and has the same scaling behavior for the approximation error as the best polynomial approximation.
- Designed an efficient circuit for QSP with respect to a black-box functions. Applies to known problems such as Hamiltonian simulation ($f(x) = e^{itx}$), matrix inversion ($f(x) = 1/x$), time-independent linear differential equation solver ($f_1(x) = e^{itx}$, $f_2(x) = (1 - e^{itx})/x$) etc.
- Achieves nearly optimal total complexity *including classical operations* if f is continuous with Lipschitz constant in $O(1)$.

Algorithm	Queries to O_f	Queries to U_H	Classical operations	Error
Remez exchange + Deterministic angle finding	∞	$\mathbf{O}(d)$	$\mathbf{O}(d^2)$	$E_d(f)$
Chebyshev truncation + Deterministic angle finding	$\text{poly}(1/\epsilon)$	$\mathbf{O}(d)$	$\mathbf{O}(d^2)$	$\tilde{\mathbf{O}}(E_d(f))$
Chebyshev interpolation + Deterministic angle finding	$\mathbf{O}(d)$	$\mathbf{O}(d)$	$\mathbf{O}(d^2)$	$\tilde{\mathbf{O}}(E_d(f) + \epsilon)$
Chebyshev truncation + Optimization	d	$\mathbf{O}(d)$	Unknown	$\tilde{\mathbf{O}}(E_d(f))$
This work	2	$\mathbf{O}(d)$	$\text{polylog}(d, 1/\epsilon)$	$(1+\sqrt{2})E_d(f) + \epsilon$

References

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