Randomized estimators of the Hafnian of a non-negative matrix

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Motivation

- Gaussian Boson Sampling effectively samples submatrices of a given matrix according to their Hafnians
- Calculating the Hafnian is #P-hard, but there are probabilistic estimators for non-negative matrices
- Depending on statistical properties, these estimators could "dequantize" some proposed applicaitons of GBS

Gaussian Boson Sampling

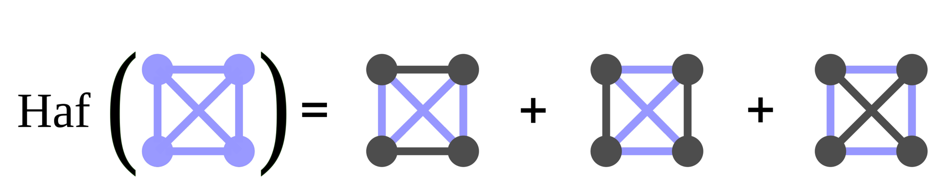
A perfect Gaussian boson sampler with M modes is specified by m squeeze parameters $r_i \in \mathbb{R}$ and the interferometer unitary $U \in \mathcal{U}(M)$. The **kernel matrix** $\mathcal{A}$ is then defined as $A \oplus A^*$, where

$$A = U\,\mathrm{diag}(\tanh r_1 \ \ldots \ \tanh r_M)U^T. \quad (1)$$

A measurement outcome $\mathbf{n} = (n_1, \ldots, n_M)$, where n_j is the number of photons measured in mode j, corresponds to a submatrix $A_{\mathbf{n}}$ of $\mathcal{A}$ where j-th and $(j+m)$-th row and column are taken n_j times. The probability of observing such an outcome is proportional to

$$P(\mathbf{n}) \propto \frac{1}{n_1! \ldots n_M!}\mathrm{Haf}(\mathcal{A}_{\mathbf{n}}). \quad (2)$$

Here the **Hafnian** function treats a matrix as an adjacency matrix of a graph and counts its **perfect matchings**:



For edge-weighted graphs, a perfect matching contributes a product of its edge weights.
Denser graphs tend to have more perfect matchings, so GBS can be used as a heuristic to find cliques in a graph.

Randomized estimators

Assuming that $A \in \mathbb{R}^{2m\times 2m}$ is a non-negative matrix (i.e. all $a_{ij} \geq 0$), we can construct an estimator of the Hafnian. Let $W \in \mathbb{R}^{2m\times 2m}$ be a random skew-symmetric matrix such that $\mathbb{E}w_{ij} = 0$, $\mathbb{E}w_{ij}^2 = 1$, and all above-diagonal entries are i.i.d. Define G to be a matrix with $g_{ij} = w_{ij}\sqrt{a_{ij}}$. Then $\mathbb{E}\det G = \mathrm{Haf}\,A$.
Here we look at two estimators:

- **Barvinok estimator**: $w_{ij} \sim \mathcal{N}(0,1)$.
- **Godsil-Gutman estimator**: $w_{ij} \in \{-1, 1\}$, sampled with equal probability.

Numerical results for Erdős-Rényi graphs

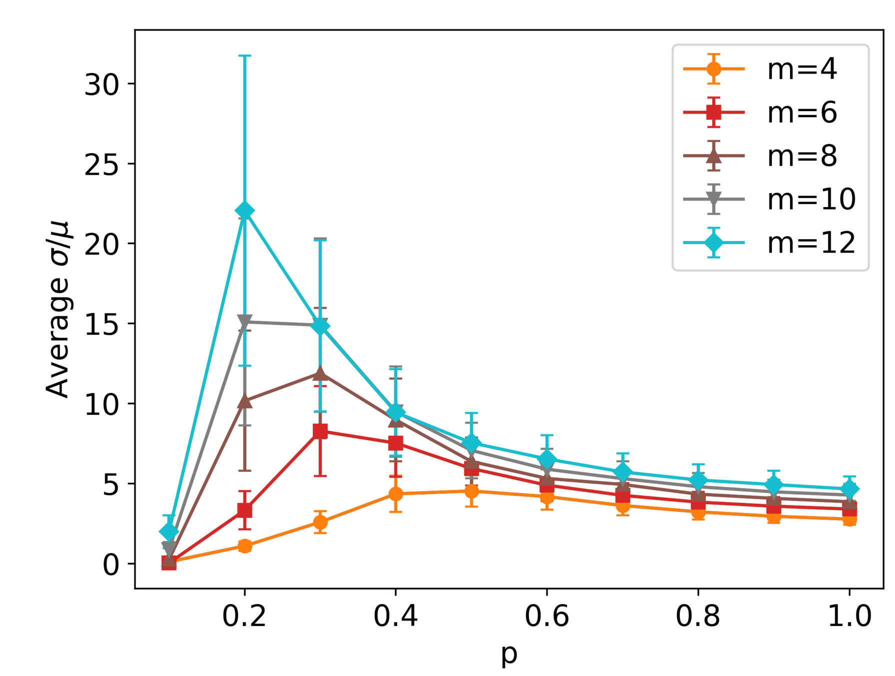


Figure 1:Relative deviation for the Barvinok estimator.

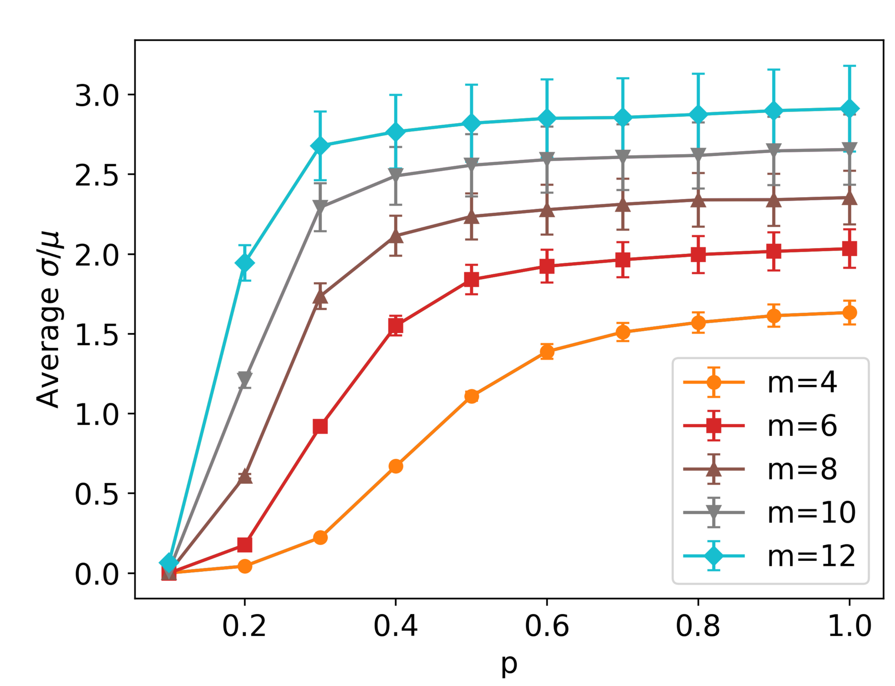


Figure 2:Relative deviation for the Godsil-Gutman estimator.

Analytical results

The variance can be expressed in terms of **perfect 2-matchings**, i.e. spanning subgraphs such that all connected components are either cycles or isolated edges:

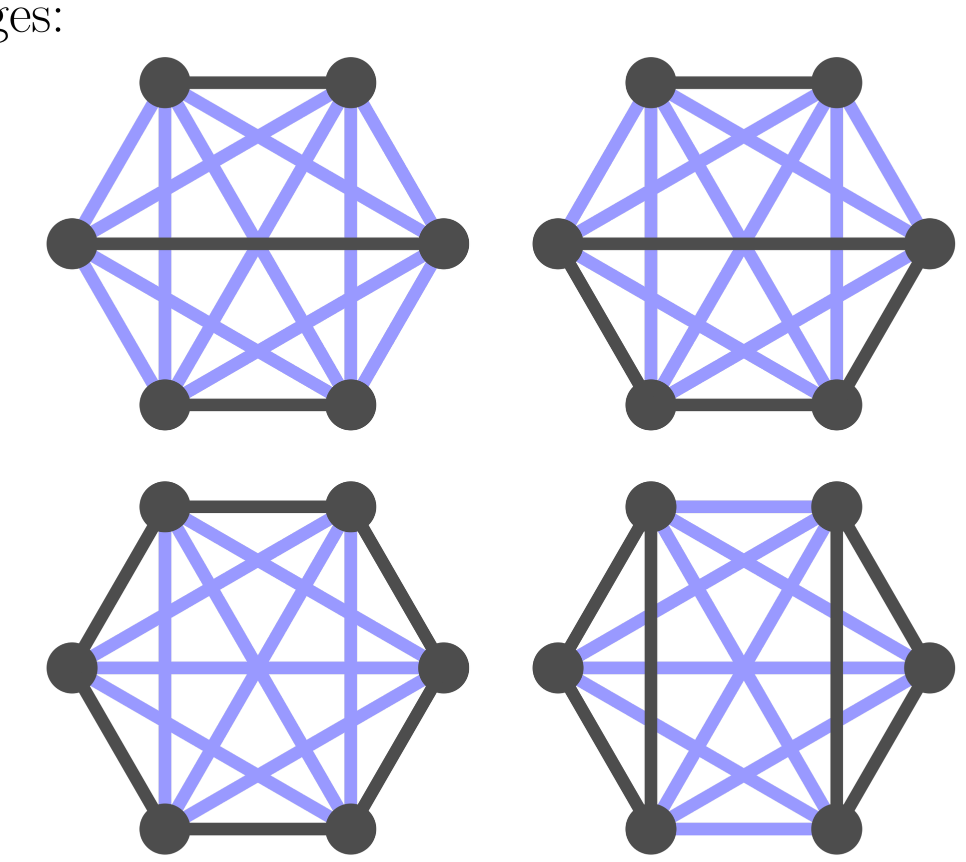


Figure 3:Examples of perfect 2-matchings. The one on the bottom right contains cycles of odd length; such 2-matchings do not contribute to the variance.

Proposition 1. *Let* $\mathbb{E}w_{ij}^3 = 0, \mathbb{E}w_{ij}^4 = \eta$. *Then*

$$\mathbb{E}(\det G)^2 = \sum_d \eta^{|\mathrm{match}(d)|} 6^{|\mathrm{cycle}(d)|} \prod_{\substack{\{i,j\}\in\mathrm{match}(d)\\ \{k,l\}\in d\setminus\mathrm{match}(d)}} a_{ij}^2 a_{kl}.$$

Here the sum is taken over all perfect 2-macthings d that contain cycles of even length; match(d) *is the set of isolated edges in d;* cycle(d) *is the set of even-length cycles in d.*
For Barvinok estimator $\eta = 3$, while for Godsil-Gutman estimator $\eta = 1$.

Proposition 2.

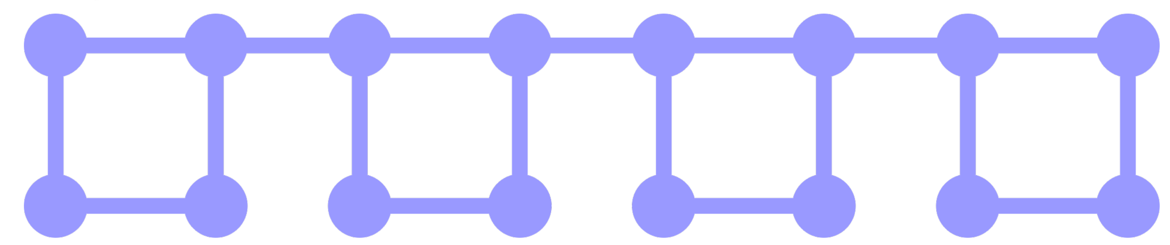
$$(\mathrm{Haf}\,A)^2 = \sum_d 2^{|\mathrm{cycle}(d)|} \prod_{\substack{(i,j)\in\mathrm{match}(d)\\ (k,l)\in d\setminus\mathrm{match}(d)}} a_{ij}^2 a_{kl}. \quad (3)$$

Theorem 1. *Let A be the adjacency matrix of a complete graph with $2m$ vertices. Then*

$$\frac{\mathbb{E}\det G^2}{(\mathrm{Haf}\,A)^2} = \sqrt{\pi m}\, e^{\frac{\eta-3}{2}} + O(1), \ m \to \infty. \quad (4)$$

Special cases

A graph like this will require an exponential number of samples to get the Hafnian with a constant accuracy:



Examples like this can be cracked with the FKT algorithm which calculates the Hafnian for planar graphs in polynomial time. However, adding a few more edges to make the graph non-planar will make FKT useless as well.

Conclusions

- We investigated the statistical properties of the Hafnian estimators and found that the Hafnian of a random graph is typically easy to estimate.
- We derived the expression for the estimator variance in terms of perfect 2-matchings. The Godsil-Gutman estimator always has a smaller variance than the Barvinok estimator.
- We prove that both esimators demonstrate a linear scaling of relative variance for for complete graphs.

Open question: is there a fully polynomial randomized approximation scheme (FPRAS) for the Hafnian? If yes, GBS experiments based on random graphs would be classically simulable.

Full article

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