Performance of randomized estimators of the Hafnian of a nonnegative matrix

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Counting the number of perfect matchings in a graph is a well-known computationally hard problem. The function that counts the perfect matchings, known as the Hafnian, appears in consideration of a model of quantum computation called Gaussian Boson Sampling (GBS). In this model, a particular optical setup is used to produce samples from a probability distribution which is believed to be hard to reproduce classically.

A proposed route to applications of GBS consists in using the sampler as an approximate solver for the densest subgraph problem and related problems. The kernel matrix of the Gaussian state is taken to be a rescaled adjacency matrix of the problem graph. That way, the detection events correspond to vertex subsets and their induced subgraphs. The probability of observing a particular event is proportional to the Hafnian of the induced subgraph, which is correlated with its density. Thus, the most likely samples are approximate solutions of the densest subgraph problem, or at least good starting points for a classical search algorithm. Crucially, in all graph-related applications to date, the kernel matrix is nonnegative.

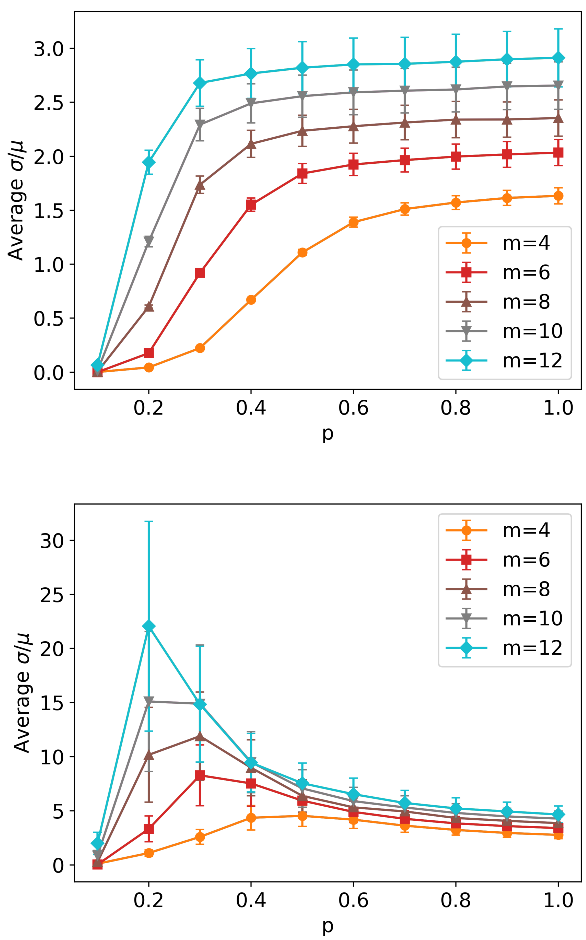

FIG. 1: Average relative standard deviation σ/μ of (a) Godsil-Gutman and (b) Barvinok estimators for random graphs.

This special case has recently been considered as potentially accessible for classical simulation. The key element of the classical simulation is a probabilistic method of estimating the Hafnian of a nonnegative matrix due to Barvinok. The mean of the estimator μ is guaranteed to be equal to the Hafnian, but the variance σ^2 affects the size of the sample that is needed to achieve a small relative error.

In this work, we investigate the behavior of the Barvinok estimator and a related method known as the Godsil-Gutman estimator. Our main findings are the following: First, we provide an analytical formula for σ/μ as a sum over certain graph coverings known as perfect 2-matchings. We show that the variance of the Godsil-Gutman estimator is always smaller or equal to that of the Barvinok estimator. We also show that, for certain classes of matrices, the relative variance σ/μ scales exponentially with the size of the problem. In addition, we provide a new asymptotic estimate of σ for the case when the input matrix is the adjacency matrix of the complete graph.

Second, we show numerically that for adjacency matrices of random graphs sampled from the Erdös-Rényi ensemble, the Godsil-Gutman estimator demonstrates a modest growth of variance across the whole range of graph densities. Specifically, for the complete graph, σ/μ grows as a square root of the graph size, and for all other densities, the average σ/μ is smaller than that.

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[1] Alexey Uvarov and Dmitry Vinichenko. Performance of randomized estimators of the Hafnian of a non-negative matrix. *Physical Review A*, 109(4):042415, April 2024.