Representation Theory for Quantum Computing

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Outline

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Hope you enjoy it!

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We believe polynomial-sized **quantum circuits to be strictly more powerful** than polynomial-sized **classical circuits**, and attend quantum computing conferences to argue about potential quantum speedups and applications.

Quantum Computing = Linear Algebra ?

Quantum states are vectors $|\psi\rangle \in \mathcal{H} = \mathbb{C}^d$ with $d = 2^n$, and quantum circuits are $d \times d$ unitary matrices. Then

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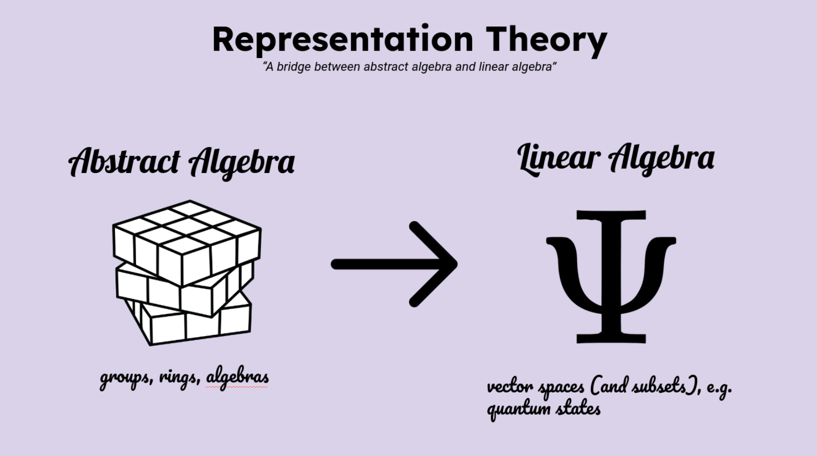
Well, yes and no. In general, we dont care for all high-dimensional linear transformations, but instead for certain **very-structured subsets** that arise from non-linear-**algebraic structures** (groups, rings, algebras) **embedded into linear transformations**.

What and Why's of Rep Theory



Rep Theory Preliminaries: Groups

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2. Every element has an inverse

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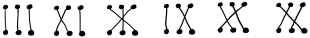
Given a set X, **the symmetric group** S_X is the set of bijections from X to X. If $|X| = n$ there are $n!$ bijections; we typically denote it S_n. For example,

$$S_3 = \Big\{(), (12), (13), (23), (123), (132)\Big\}.$$

We use notation $() = (1)(2)(3)$ or $(12) = (12)(3)$ in which we omit one-cyles.



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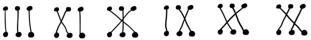

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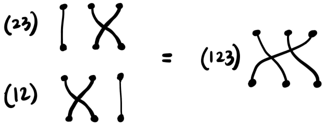

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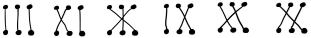

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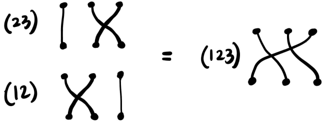

Other examples of **finite groups** are the more general **permutation groups**, the subgroups $G \subseteq S_n$. Examples include the **alternating group** A_n of even permutations, or the **cyclic group** $\mathbb{Z}_n = \langle(12 \cdots n)\rangle \subset S_n$ generated by an n-cycle.

Rep Theory Preliminaries: Groups

Another example of a group, in this case infinite, is the **Unitary Group**[2] $\mathrm{U}(d)$ of $d \times d$ unitary matrices[3]. Similarly, the **Special Unitary Group** $\mathrm{SU}(d)$ corresponds to the subgroup satisfying $\det(U) = 1$.

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$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset \mathrm{SU}(2).$$

and $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{U}(2)$ and $\notin \mathrm{SU}(2)$ because $\det(U) = -1$.

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As before, check that these are groups. For example, $iX \circ iZ = iY$ and iY which is $\in \mathrm{SU}(2)$ (since $(iY)(iY)^\dagger = (iY)(-iY) = I$)

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A **unitary** (group) **representation** R of G on V is **a map**

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that is a **homomorphism** between the groups G and $\mathrm{U}(V)$, in the sense that it preserves the group structure:

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A vector space V that supports a G-rep is called a G-**module**.

Rep Theory Preliminaries: Examples of reps

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Symmetric Group (and subgroups): Let $G \subseteq S_n$.

1. **Permutation representation:** the action of G on n-dimensional vector space $V = \mathbb{C}^n = \mathrm{span}\{|i\rangle\}_{i=1}^n$ by permutation of canonical basis,

$$R_{\mathrm{def}}(\sigma)\,|i\rangle = |\sigma(i)\rangle$$

2. **Regular representation:** the action of G on $V = \mathbb{C}^{|G|} = \mathrm{span}\{|g\rangle\}_{g \in G}$ (that is, on itself) by

$$R_{\mathrm{reg}}^{\mathrm{left}}(g)\,|h\rangle = |gh\rangle$$

or

$$R_{\mathrm{reg}}^{\mathrm{right}}(g)\,|h\rangle = |hg^{-1}\rangle$$

Fun fact: Group representations describe **symmetries**, transformations that leave certain vectors **invariant**. Think of the S_n-invariance of GHZ state, or the SU(2)-invariance of isotropic (XXX) Heisenberg interactions.

Irreducible Representations and Decompositions

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For example, consider the permutation representation of S_4 on $V = \mathbb{C}^4$, where $R(\sigma)\,|i\rangle = |\sigma(i)\rangle$. Some explicit elements in the image of R are

$$R\left((1,2)\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R\left((1,2,3,4)\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Q: Is this irreducible?

A: No, there is a subspace $W = \mathrm{span}\left\{ \sum_{i=1}^{4} |i\rangle \right\}$ that is invariant under all permutations. But if we express $V = W \oplus W^{\perp}$, with $W^{\perp} = \mathrm{span}\{|1\rangle - |2\rangle, |2\rangle - |3\rangle, |3\rangle - |4\rangle\}$ the (3-dimensional) complement of W, we can verify that these two are irreducible.

Complete Reducibility

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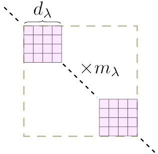
Explicitly, this means there is a change of basis U that takes V to a direct sum of irreducible submodules $V_G^{\alpha,i} \subset V$, where α is used to label the different irrep instances and $i \in m_\alpha$ is a copy (or multiplicity) index. In this basis, the group action is **block-diagonal**, with indentical blocks $r_G^\alpha(g)$ of size $\dim(V^\alpha) \equiv d_\alpha$ repeated m_α times:

$$UR(g)U^\dagger = \bigoplus_\alpha I_{m_\alpha} \otimes r_G^\alpha(g) \tag{2}$$

$UR(g)U^\dagger =$ 

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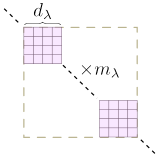

When a group is **abelian**, all irreps are one dimensional and thus, in the basis U, $R(g)$ is diagonal. For reasons that will be evident later, such basis U that **maximally block-diagonalizes** the group action is called the **Fourier basis**.

Apps of RT in QC: An outline.

I see two main areas of application of RT in QC:

- **Analitic:** RT provides a tools to analize quantum algorithms, especially but not necesarilly those containing some degree of randomization. Examples include:
  - Concentration of Variational Algorithms.
  - Classical Shadows
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- **Algorithmic:** We can use RT to develop quantum transforms and algorithms based on them. Some examples are:
  - Transforms: (abelian) Quantum Fourier Transforms (QFTs), Non-abelian QFTS, Quantum Schur Transform, etc.
  - Algorithms based on such transforms, include phase estimation, (abelian and non-abelian) Hidden Subgroup Problem, etc.

A first app: the Discrete Fourier Transform (DFT) and the abelian QFT

Consider the **cyclic group** $\mathbb{Z}_N$ and its representation shifting the computational basis of $V = \mathbb{C}^N$, explicitly given by

$$R(j) = \sum_{i=1}^{N} |i\rangle\langle i+j|$$

The action of $j \in \mathbb{Z}_N$ on $\mathbb{C}^N$ is to translate its basis by j units.

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Consider a function $f : \mathbb{Z}_N \to \mathbb{C}$, or equivalently a vector $|f\rangle = \sum_{i=1}^{N} f(i)\,|i\rangle$. The **Discrete Fourier Transform** (DFT) maps 'discrete position basis'

$$|i\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k\in[N]} w_N^{ik}\,|k\rangle$$

to 'discrete momentum basis', where $w_N = e^{i2\pi/N}$ is the N-th root of unity.

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Each $|k\rangle$ spans a one-dimensional non-isomorphic irreducible representation[4] $V_{\mathbb{Z}_N}^{k} = \mathrm{span}\{|k\rangle\}$, such that

$$V \cong \bigoplus_{k\in[N]} V_{\mathbb{Z}_N}^{k} \tag{3}$$

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FFT and QFT

The cost of performing DFT is $\mathcal{O}(N^2)$[5]. The DFT is unitary, and can be compiled into a quantum circuit (encoding $\mathbb{C}^N$ with $\log(N)$ qubits) using only $\tilde{\mathcal{O}}(\log N)$ gates[6] –welcome to the **Quantum Fourier transform** (QFT)[7].

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This quantum speedup in rotating discrete position to basis where cyclic group acts diagonally has led to some of the most successful quantum algorithms known to date:
Period Finding (PF):

- Given a function $f : \mathbb{Z}_N \to S$, promised to be periodic: $\exists s$ such that $f(x) = f(s + x)$, find the period s.
- Classical algorithms take $\mathcal{O}(s)$ time; s could be of order $N = 2^n$. Shor's quantum algorithm is efficient, works in $\tilde{\mathcal{O}}(n^2)$ time.
- PF is an instance of the more general problem called the Abelian Hidden Subgroup Problem (HSP).

Factoring:

- Given N, find unique $\{m_i\}$ such that $N = 2^{m_1} 3^{m_2} \cdots$.
- For $N = 2^n$, best classical algorithms run in time $\mathcal{O}(\exp(\sqrt{n}))$. Instead, Shor's quantum algorithm (an application of PF) runs in time $\tilde{\mathcal{O}}(n^2)$.

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What's special about $G = \mathbb{Z}_{2^n}$? re: not much! We will come back to this in the more general setting of other $G \subseteq S_n$ than $\mathbb{Z}_n$ and the diagonalization of their regular reps – we will call efficient compilations of them into quantum circuits '*G*-**QFTs**'.

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RT of the symmetric group

In general, the irreducible representations of a group are in one-to-one correspondence with their **Conjugacy Classes** (CCs). As we'll see below, S_n conjugacy classes are parametrized by integer partitions $\lambda \vdash n$, and thus, so will S_n-irreps be.

[8] A sequence $\lambda = (\lambda_1, \cdots, \lambda_n)$ of weakly decreasing positive integers $\lambda_1 \geq \cdots \geq \lambda_n$) that add up to n. Note that this condition restricts to left-justified Young diagrams.

[9] A collection of boxes, arranged in left-justified rows with row-lengths in non-increasing order.

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$$\sigma = (1,3,5)(2,4)(6)(7,8) \in S_8$$

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Since cycles are disjoint, order doesnt matter so we can order larger cycles first. Thus any given **cycle type** can be captured by a partition[8] $\lambda \vdash n$, a way of breaking down n into (at most n) parts. For example, we here have $8 = 3 + 2 + 2 + 1$, corresponding to the partition $\lambda = [3, 2, 2, 1] \vdash 8$.

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Since cycles are disjoint, order doesnt matter so we can order larger cycles first. Thus any given **cycle type** can be captured by a partition[8] $\lambda \vdash n$, a way of breaking down n into (at most n) parts. For example, we here have $8 = 3+2+2+1$, corresponding to the partition $\lambda = [3,2,2,1] \vdash 8$.

Partitions are typically depicted diagramattically by **Young diagrams**[9]

$\lambda = [4] \quad \leftrightarrow$ 

$\lambda = [3,1] \quad \leftrightarrow$ 

$\lambda = [2,1,1] \quad \leftrightarrow$ 

$\lambda = [2,2] \quad \leftrightarrow$ 

$\lambda = [1,1,1,1] \quad \leftrightarrow$ 

[8] A sequence $\lambda = (\lambda_1, \cdots, \lambda_n)$ of weakly decreasing positive integers $\lambda_1 \geq \cdots \geq \lambda_n$) that add up to n. Note that this condition restricts to left-justified Young diagrams.

[9] A collection of boxes, arranged in left-justified rows with row-lengths in non-increasing order.

RT of the symmetric group

Consider the act of conjugating σ by some other permutation π: while σ changes into $\sigma' = \pi\sigma\pi^{-1}$, its cycle type doesn't. The **conjugacy class** (CC) of some σ in S_n is the subset

$$C_\sigma = \left\{\pi\sigma\pi^{-1} \,|\, \pi \in S_n\right\} \tag{4}$$

Let λ be the cycle type of σ. It turns out that $C_\sigma = C_\lambda = \{\pi \in S_n \text{ with c.t. } \lambda.\}$.

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CCs partition a group into disjoint subsets $S_n = \bigcup_{\lambda \vdash n} C_\lambda$. For example

$$S_3 = \underbrace{\left\{()\right\}}_{C_{\ydiagram{1,1,1}}} \bigcup \underbrace{\left\{(12), (13), (23)\right\}}_{C_{\ydiagram{2,1}}} \bigcup \underbrace{\left\{(123), (132)\right\}}_{C_{\ydiagram{3}}} \tag{5}$$

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Having estabilshed that S_n-CCs and thus S_n-irreps are labelled by partitions $\lambda \vdash n$, we turn to their construction. We will denote S_n-irrep labelled by some $\lambda \vdash n$, $V^{\lambda}_{S_n}$.

Young's Rule and Specht Modules

Consider $V^{\lambda}_{S_n}$ as a rep of the subgroup $S_{n-1} = \{\sigma \in S_n \,|\, \sigma(n) = n\} \subset S_n$. Example here[10].

[10] For example, $\sigma = (12)(34) \in S_4$ doesnt fix 4, but $\sigma = (123)(4)$ does.

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$$V^{\lambda}_{S_n} \downarrow_{S_{n-1}} \cong \bigoplus_{\lambda_{n-1} \in \lambda - \square} V^{\lambda_{n-1}}_{S_{n-1}} \tag{6}$$

where $\lambda_{n-1} \vdash n-1$. Crucially, we can iterate this process until we reach $V^{\square}_{S_1}$.

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$$T \equiv (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \cdots, \lambda) \tag{7}$$

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We can more compactly describe T by a diagram λ that is filled the set $[n]$ in such a way that the filling encodes the path. For example, consider a path $T = (\yng(1), \yng(1,1), \yng(2,1), \yng(2,2))$. The corresponding **Standard Young Tableau** (SYT)

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Instead the path $T' = (\square, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,2}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$.

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Young's Basis

We know λ labels some S_n-irrep $V_{S_n}^{\lambda}$, but what is its dimension? We can use Young's rule to figure that out:

$$\dim(V_{S_n}^{\lambda}) = \dim\left(\bigoplus_{\lambda_{n-1}} \cdots \bigoplus_{\lambda_2} \bigoplus V_{S_1}^{\square}\right) = \sum_{T \in \mathrm{SYT}(\lambda)} = |\mathrm{SYT}(\lambda)| \,. \tag{9}$$

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If the dimension of λ equals the number of SYTs, we can use each SYT to label a distinct linearly independent basis vector in $V_{S_n}^{\lambda}$. We get **Young's basis**

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All that remains to finish working out the representations $V^{\lambda}_{S_n}$ is to understand how the group acts on this basis of SYTs.

Young's Orthogonal Representation: Action of S_n on SYTs

Since S_n is generated by adjacent transpositions $t_i = (i, i+1)$

$$\left\langle \left\{ t_i \right\}_{i=1}^{n-1} \right\rangle = S_n \tag{11}$$

it is sufficient to work out the action of each t_i on the SYTs.

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$$t_i \cdot |T\rangle = \frac{1}{\Delta_i(T)} |T\rangle + \sqrt{1 - \frac{1}{\Delta_i(T)^2}} |t_i \cdot T\rangle \tag{12}$$

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- The content of a cell $u = (r, c)$ is just $\mathrm{cont}(u) = c - r$

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Young's Orthogonal Representation: Example.

For example, lets find the rep matrix for $t_1 = (12) \in S_3$ on $V_{S_3}^{\ydiagram{2,1}} = \mathrm{span}\{|\ytableaushort{12,3}\rangle, |\ytableaushort{13,2}\rangle\}$.

[12] It turns out the rep matrices are orthogonal.

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If we also work out $r_{S_3}^{\ydiagram{2,1}}((23))$, then given any $\sigma \in S_3$ we can compile [12] [13] it into a product of such adjacent transpositions[14] $\sigma = \prod_{a=1}^{N} g_a$ with $g_a \in \{t_1, t_2\}_i$, and finally find $r_{S_3}^{\ydiagram{2,1}}(\sigma)$ by composing such rep matrices for the generators $\prod_{i=1}^{N} r_{S_3}^{\ydiagram{2,1}}(t_i)$.

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Fourier Analysis on Groups and G-QFT

Consider some finite group $G \subseteq S_n$, and vectors $|f\rangle = \sum_{g \in G} f(g) |g\rangle \in \mathcal{H}$ (left-regular representation).

The G-**QFT** is the unitary transformation that block-diagonalizes such group action[15]. For certain groups, this transform can be compiled into **efficient** quantum circuit:

- $G = \mathbb{Z}_{2^n}$, the 'vanilla' QFT is efficient (as we saw previously).
- $G = S_n$ and certain subgroups[16] are efficient.

[15] by mapping the group basis $\{|g\rangle\}$ to the irrep basis $\{|\lambda, i, j\rangle\}$, where λ labels irreducible representations, and i, j index the multiplicity and dimension.

[16] works by Beals and Moore. Note that $\log(n!) \sim n \log(n)$ so $\tilde{\mathcal{O}}(n)$ qubits for the register. The circuit compilations are of depth $\tilde{\mathcal{O}}(n^2)$, like abelian case.

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What **can we do** with these non-abelian G-QFTs?
In [LH24] we show that one can '**factor representations**' – instead of finding $\boldsymbol{m}$ such that $N = 2^{m_1} 3^{m_2} \cdots$ we find $\boldsymbol{m}$ such that $V = V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots$:

Table I: **Quantum Algorithms for Branching Coefficients.** Choices of groups $H \subseteq G$, H-representation R, and an H-irrep r_H^α for which the algorithm in Thm. 1 computes the studied multiplicity coefficients. Here, $n, a, b, c, d \in \mathbb{N}$ where $a + b = n$ (Littlewood-Richardson) and $cd = n$ (Plethysm). The cost $\dim(R)/\dim(r_H^\alpha)$ is the number of samples required to (exactly) compute the coefficient .

Problem	Input	Group G	Subgroup $H \subseteq G$	H-Rep R	H-Irrep r_H^α	Output $\mathbf{mult}(r_H^\alpha, R)$	Cost $\mathcal{O}(\frac{\dim(R)}{\dim(r_H^\alpha)})$
Kostka	$\nu, \mu \vdash n$	S_n	$S_\mu := \bigtimes_i S_{\mu_i}$	$(r_{S_n}^\nu) \downarrow_{S_\mu}^{S_n}$	$\bigotimes_i r_{S_{\mu_i}}^{(\mu_i)}$	K_ν^μ	$\mathcal{O}(d_\nu)$
Littlewood	$\nu \vdash n$ and $\lambda, \mu \vdash a, b$	S_n	$S_a \times S_b$	$(r_{S_n}^\nu) \downarrow_{S_a \times S_b}^{S_n}$	$r_{S_a}^\lambda \otimes r_{S_b}^\mu$	$c_{\lambda\mu}^\nu$	$\mathcal{O}(\frac{d_\nu}{d_\lambda d_\mu})$
Plethysm	$\nu \vdash n$ and $\lambda, \mu \vdash c, d$	S_n	$S_c \wr S_d$	$(r_{S_n}^\nu) \downarrow_{S_c \wr S_d}^{S_n}$	$r_{S_c}^\lambda \wr r_{S_d}^\mu$	$a_{\lambda\mu}^\nu$	$\mathcal{O}(\frac{d_\nu}{d_\lambda^b d_\mu})$
Kronecker [1]	$\nu, \lambda, \mu \vdash n$	$S_n \times S_n$	S_n	$(r_{S_n}^\lambda \otimes r_{S_n}^\mu) \downarrow_{S_n}^{S_n \times S_n}$	$r_{S_n}^\nu$	$g_{\lambda\mu\nu}$	$\mathcal{O}(\frac{d_\lambda d_\mu}{d_\nu})$

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Representations of the Unitary group

Consider SU(d) and let $V = \mathbb{C}^d$ the standard representation. If we take

$$V^{\otimes 2} \cong \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V) \tag{14}$$

of dims $d(d+1)/2$ and $d(d-1)/2$. Both are SU(d)-irreps.

[17] Meaning entries are poly of the entries of the standard rep matrix elements. Instead, rational reps are rational functions of the matrix elements, e.g. the determinant rep.

[18] spoiler: appears $\dim(V_{S_n}^{\lambda})$ times.

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of dims $d(d+1)/2$ and $d(d-1)/2$. Both are $\mathrm{SU}(d)$-irreps.

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Representations of the Unitary group

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Given $\lambda \vdash n$, irrep $V^{\lambda}_{\mathrm{SU}(d)}$ appears (at least once[18]) in $V^{\otimes n}$.

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Gelfand-Tseltsin basis

Not only we can label SU(d) irreps by λ, but also, just like for S_n, their branching is **multiplicity free**

$$V^{\lambda}_{SU(d)} \downarrow_{SU(d-1)} \cong \bigoplus_{\substack{\lambda_{d-1} \in \lambda - \square \\ l(\lambda_{d-1}) \leq d-1}} V^{\lambda_{d-1}}_{SU(d-1)} \tag{15}$$

Thus, again, we can **label basis by paths** in the restriction[19].

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The condition of removing boxes given the length constraint can be restated graphically by aligning the vectors λ_k in an inverted pyramid called **GT pattern** and asking each of the entries of the vectors to lie **in between** the two entries just above[21]

$$M = \begin{bmatrix} \lambda_{d,1} & & \lambda_{d,2} & & \cdots & & \lambda_{d,d-1} & & \lambda_{d,d} \\ & \lambda_{d-1,1} & & & \cdots & & & \lambda_{d-1,d-1} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & \lambda_{2,1} & & \lambda_{2,2} & & & \\ & & & & \lambda_{1,1} & & & & \end{bmatrix}$$

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GT patterns and SU-irreps: Examples.

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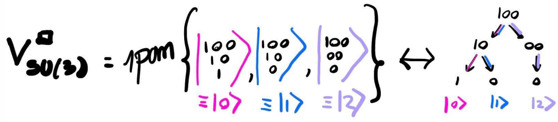

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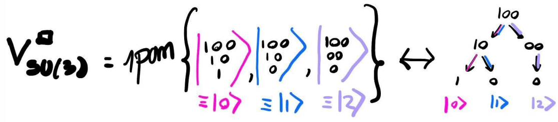

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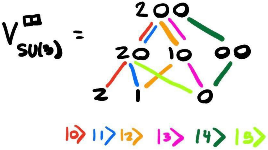

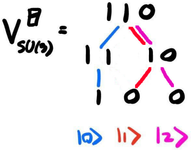

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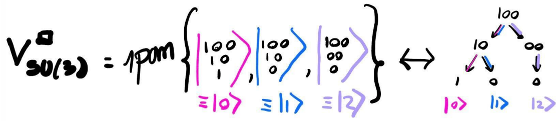

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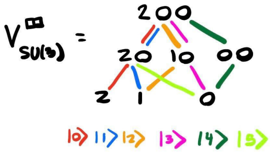

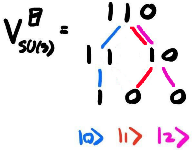

Finally, again, this route takes us to explicit matrix construction[22].

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The Schur-Weyl duality: Linear Algebraic version

We've learnt about SU(d) and S_n RTs. It turns out they are intertwined.

[23]Given a G-module V, denote V^G the subspace of G-invariants (aka, the **commutant** of G). Similarly, $V^{\mathcal{A}}$ is $\mathcal{A}$-invariant subspace of $\mathcal{A}$-module V for $\mathcal{A}$ some matrix algebra.

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Note that these actions **commute**. For all $U \in \mathrm{SU}(d)$ and $\sigma \in S_n$, $[Q(U), P(\sigma)] = 0$. But there's more to it. Let us introduce two matrix subalgebras of $\mathrm{End}(\mathcal{H})$[23]

$$\mathcal{U} = \mathrm{span}\{Q(U)\}_{U \in \mathrm{SU}(d)} \text{ and } \mathcal{P} = \mathrm{span}\{P(\sigma)\}_{\sigma \in S_n} \tag{16}$$

Then,

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This is the **linear algebraic version** of SW duality: not only $P(\sigma)$ commutes with all $U^{\otimes n}$, but **all that commutes with $U^{\otimes n}$ is a sum of permutations**:

$$[M, U^{\otimes n}] = 0 \leftrightarrow M = \sum_{\sigma} c_\sigma P(\sigma)\,. \tag{18}$$

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Apps of SW duality: Computing Moments of unitary designs

A clear application of SW duality is in computing moments of Haar random unitaries or designs.

[24] Here, $\chi_\alpha(g) = \mathrm{Tr}[R^\alpha(g)]$ is the character of irrep α. Similar expression for Lie groups replaces the uniform probability $p(g) = \frac{1}{|G|}$ by the Haar measure $p_{\mathrm{Haar}}(U)$.

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Let $\mathcal{H} = (\mathbb{C}^d)^{\otimes t}$ and $\mathcal{L}(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. Let $\mathcal{T}^{(t)}_{\mathsf{SU}(d)} = \int_{U \in \mathsf{SU}(d)} U^{\otimes t}(\cdot)(U^\dagger)^{\otimes t}$ be the t-th fold **moment superoperator** (or **twirl**) from $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, and $T^{(t)}_{\mathsf{SU}(d)} = \mathrm{vec}\mathcal{T}^{(t)}_{\mathsf{SU}(d)} = \mathbb{E}_U[U^{\otimes t} \otimes \bar{U}^{\otimes t}]$ its vectorization.

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It is a well known fact that given a rep R of group G, one can explicitly write a projector into any isotypic α via

$$\Pi_\alpha = \frac{d_\alpha}{|G|} \sum_{g \in G} \chi_\lambda(g) R(g) \tag{19}$$

Instantiation[24] of this expression for $G = \mathsf{SU}(d)$, $R(U) = U^{\otimes t} \otimes \bar{U}^{\otimes t}$ and $\alpha = \mathrm{triv}$ (with $\chi_\alpha(U) = 1$) takes us to $T^{(t)}_G$, meaning $T^{(t)}_{\mathsf{SU}(d)}$ is the orthogonal projector onto $\mathcal{L}(\mathcal{H})^{\mathsf{SU}(d)}$, the trivial isotypic of the $\mathsf{SU}(d)$-module $\mathcal{L}(\mathcal{H})$ – the **commutant** of $U^{\otimes t} \otimes \bar{U}^{\otimes t}$.

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Moment Computations

Since $T^{(t)}_{\mathsf{SU}(d)}$ is a projector onto commutant $\mathcal{L}(\mathcal{H})^{\mathsf{SU}(d)}$ and SW provides an explicit (non-orthogonal) basis

$$\mathcal{L}(\mathcal{H})^{\mathsf{SU}(d)} = \mathrm{span}\{P(\sigma)\}_{\sigma \in S_t} \tag{20}$$

in terms of $t!$ permutations, we can use this to expand[25]

$$T^{(t)}_{\mathsf{SU}(d)} = \frac{1}{d^t} \sum_{\sigma,\pi \in S_t} W_{\sigma,\pi} |P(\pi)\rangle\!\rangle\langle\!\langle P(\sigma)| \tag{21}$$

where $W_{\sigma,\pi}$ are coefficients that come from inversion of the Gram matrix of $\{P(\sigma)\}$ basis[26]. This approach goes by the fancy name of **Weingarten Calculus**, but its just using a duality to expand the commutant.

Some applications of moment computation:

1. $t = 2$: Barren Plateau.
2. $t = 3$: Classical Shadows.
3. $t \geq 3$: Gaussian Processes.

---

[25] We write a vectorized $O \in \mathcal{L}(\mathcal{H})$ as $|O\rangle\!\rangle$.

[26] For example, in the case $t = 2$ the Weingarten matrix W (Gram's inverse, such that $W_{\pi\sigma} = [W]_{\pi,\sigma}$) is $W = \frac{1}{d^2-1}\begin{pmatrix} 1 & -1/d \\ -1/d & 1 \end{pmatrix}$ for $\{P(()), P((12))\}$.

Barren Plateau

For example, let $l_U = \langle\langle O \big| U \big| \rho \rangle\rangle$ be the cost function of a VQA and suppose we initialize random parameters that roughly correspond to U sampled from a 2-design in $\mathrm{SU}(d)$. We can compute the variance

$$\mathrm{Var}_U[l_U] \leq \mathbb{E}_U[l_U^2] = \langle\langle O^{\otimes 2} \big| T^{(t)}_{\mathrm{SU}(d)} \big| \rho^{\otimes 2} \rangle\rangle \tag{22}$$

$$\sim \frac{1}{d^2} \sum_{\sigma \in S_2} \langle\langle O^{\otimes 2} | P(\sigma) \rangle\rangle \langle\langle P(\sigma) | \rho^{\otimes 2} \rangle\rangle \tag{23}$$

$$= \frac{1}{d^2} \mathrm{Tr}[O^2] \mathrm{Tr}[\rho^2] \in \mathcal{O}(1/d) \tag{24}$$

Thus, we find exponential concentration of the cost function – a **barren plateau**.

The Schur-Weyl duality: Irrep version

The **irrep version** of **Schur-Weyl duality** is a statement about the representation of $(\mathbb{C}^d)^{\otimes n}$ as a $G = \mathrm{SU}(d) \times S_n$-module.

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It asserts that the only irreps of G that appear are 'diagonal' (of the form (λ, λ)), namely $V^{\lambda}_{\mathrm{SU}(d)} \otimes V^{\lambda}_{S_n}$, and they do so with multiplicity one:

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Lets absorb this. There are two natural bases for $(\mathbb{C}^d)^{\otimes n}$. On one side, the **product basis** $\{|x\rangle\}_{x \in [d]^n}$. On another, the **Schur basis**[27] $\left\{ |\lambda, T, M\rangle \right\}_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}, T, M \in \lambda}$ that **maximally block-diagonalizes** the action of G:

$$(U, \sigma) \cdot |\lambda, T, M\rangle = |\lambda, \sigma \cdot T, U \cdot M\rangle \tag{26}$$

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Lets restrict G to each of the normal subgroups:

- As a $\mathrm{SU}(d)$-module:

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- As a S_n-module:

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From irreps to commutant

Supose we got a G-module $\mathcal{H} = \mathrm{span}\{|i\rangle\}_{i=1}^{\dim(\mathcal{H})}$ and we know its decomposition is

$$\mathcal{H} \cong \bigoplus_{\alpha} \mathbb{C}^{m_\alpha} \otimes V_G^{\alpha} = \bigoplus_{\alpha} \bigoplus_{i_\alpha=1}^{m_\alpha} V_G^{\alpha,i} \tag{29}$$

We can then find the decomposition of the space of linear operators $\mathcal{L}(\mathcal{H}) \cong \mathcal{H} \otimes \bar{\mathcal{H}} = \mathrm{span}\{|i\rangle\langle j|\}_{i,j=1}^{\dim(\mathcal{H})}$, as

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A result known as **Schur's Lemma** asserts that **only way trivial irreps appear in the tensor product of two irreps** is iff the two irreps are **dual to each other** (and in such case appear with mulp one).

From irreps to commutant

Supose we got a G-module $\mathcal{H} = \mathrm{span}\{|i\rangle\}_{i=1}^{\dim(\mathcal{H})}$ and we know its decomposition is

$$\mathcal{H} \cong \bigoplus_{\alpha} \mathbb{C}^{m_\alpha} \otimes V_G^{\alpha} = \bigoplus_{\alpha} \bigoplus_{i_\alpha=1}^{m_\alpha} V_G^{\alpha,i} \tag{29}$$

We can then find the decomposition of the space of linear operators $\mathcal{L}(\mathcal{H}) \cong \mathcal{H} \otimes \bar{\mathcal{H}} = \mathrm{span}\{|i\rangle\langle j|\}_{i,j=1}^{\dim(\mathcal{H})}$, as

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Then

$$\mathcal{L}(\mathcal{H})^G \cong \bigoplus_{\alpha=\beta} \bigoplus_{i_\alpha, j_\alpha=1}^{m_\alpha} (V_G^{\alpha,i} \otimes \overline{V_G^{\alpha,j}})^G \tag{32}$$

That is, for each pair of copies of α in $\mathcal{H}$ there is a G-invariant operator $(V_G^{\alpha,i} \otimes \overline{V_G^{\alpha,j}})^G$ in $\mathcal{L}(\mathcal{H})^G$.

From irreps to commutant

Lets use irrep SW duality to get the commutant. We have

$$\mathcal{H} = (\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} V^{\lambda}_{\mathrm{SU}(d)} \otimes \mathbb{C}^{\dim(V^{\lambda}_{S_n})} \tag{33}$$

$$= \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \bigoplus_{T \in \lambda} V^{\lambda, T}_{\mathrm{SU}(d)} \tag{34}$$

where T index copies and $V^{\lambda,T}_{\mathrm{SU}(d)} = \mathrm{span}\{|\lambda, T, M\rangle\}_{M\in\lambda}$.

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$$B_{\lambda, T_1, T_2} = \frac{1}{\dim(V^{\lambda}_{\mathsf{SU}(d)})} \sum_{M \in \lambda} |\lambda, T_1, M\rangle\langle\lambda, T_2, M| \tag{35}$$

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We arrive at a new basis (now orthonormal so no need for Weingarten Calculus) for the **commutant**

$$\mathcal{L}(\mathcal{H})^{\mathrm{SU}(d)} = \mathrm{span}\{P(\sigma)\}_{\sigma \in S_n} = \mathrm{span}\{B_{\lambda,T_1,T_2}\}_{\lambda,(T_1,T_2) \in \lambda} . \tag{36}$$

BPs Again

Proceding similarly in the general case one can derive, for arbitrary dynamical Lie group G and decomposition of $\mathcal{L} = \bigoplus_\alpha \mathbb{C}^{m_\alpha} \otimes V_G^\alpha$:

$$\mathbb{E}_{U\sim G}[l_U^2] = \sum_\alpha \sum_{i,j=1}^{m_\alpha} \frac{\langle \rho_{\alpha,i}, \rho_{\alpha,j}\rangle \langle O_{\alpha,i}, O_{\alpha,j}\rangle}{\dim(V_G^\alpha)} \tag{37}$$

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In the special case $O \in \mathfrak{g}$ we have

$$\mathbb{E}_{U\sim G}[l_U^2] = \frac{\langle \rho_\mathfrak{g}, \rho_\mathfrak{g}\rangle \langle O, O\rangle}{\dim(\mathfrak{g})} \tag{38}$$

so Var is propto $1/\dim(\mathfrak{g})$ as stated in an old conjecture [Lar+21]. This proof was simultaneously derived in [Rag+23] and [Fon+23].

Apps of SW duality: Quantum Schur Transform

Consider a quantum system given by n qudit registers, $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. Suppose we want to implement a transformation mapping the computational basis $\{|x\rangle\}_{x \in [d]^n}$ to the Schur basis.

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[28] There are claims [CHW06] that a QST efficient in $\log(d)$ should be possible, and some proposals [Kro19] via induced S_n-reps.

[29] In the limit of $n >> 1$ this is optimal estimator.

[30] When all we want is to sample irrep register λ, we dont necesarilly need specific FT for given rep R of G – we sometimes can use regular G-QFT together with a controlled operation similar to the one in phase-estimation algo– known as **Generalized Phase Estimation** [BCH06]. In the case of WSS we can choose to use S_n-QFT as opposed to QST, with potential exponential-in-d savings.

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Efficient compilations of such transformation into quantum circuits have been developed – **Quantum Schur Transforms** (QST)– with complexity being $\mathrm{poly}(d, n)$ [28]. The default method proceeds by a cascade of **Clebsch-Gordan Transforms**, which essentially consists of the coupling of an arbitrary SU(d) irrep V_λ^λ with a standard one $V_{\mathrm{SU}(d)}^{\square}$.

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App: spectrum estimation [BCH06]: Given $\rho \in \mathrm{End}(\mathbb{C}^d)$ such that $\rho = \sum_i r_i |r_i\rangle\langle r_i|$, QST gives an algorithm to estimate the vector $r = (r_i)$. Initialize n copies of ρ, apply QST and measure the λ register – a routine called **weak Schur sampling** (WSS)– obtaining some partition λ with probability $p(\lambda) = \mathrm{Tr}[\rho^{\otimes n}\Pi_\lambda]$.

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Other dualities

A non-extensive enumeration of dualities includes:

1. **Orthogonal and Symplectic SW**[31]**:** States that the commutant of the t-fold standard rep of $\mathrm{O}(d)$ and $\mathrm{SP}(d)$, subgroups of $\mathsf{U}(d)$, is given by **Brauer** algebras $B_t(d)$ and $B_t(d)$, an extension of S_t algebra that allows to 'connect stuff on the same side'.
2. **Mixed SW-duality**[32]**:** Similar to SW duality but we consider the $\mathsf{SU}(d)$ rep $Q^{p,q} = U^{\otimes p} \otimes \bar{U}^{\otimes q}$. Then

$$(\mathbb{C}^d)^{\otimes p+q} \cong \bigoplus_{\gamma} V^{\gamma}_{\mathsf{SU}(d)} \otimes V^{\gamma}_{\mathcal{B}^d_{p,q}} \tag{39}$$

   where $\mathcal{B}^d_{p,q}$ is a superalgebra of S_{p+q} called the **walled Brauer algebra**.
3. **Howe duality:** for tensors powers of particle-preserving (PP) and non-PP free fermionic and free bosonic unitaries.
4. **Clifford duality:** the commutant of $U \in Cl_d \subset \mathsf{U}(d)$ tensor t is given by (i) discrete orthogonal transformations and (ii) self-orthogonal CSS code projectors[33]
5. **Regular Rep:** Left and Right regular reps are self dual, this is the reason why multiplicity of each irrep α in reg rep is its dimension! $G \cong \bigoplus_{\alpha} V^{\alpha}_G \otimes V^{\alpha}_G$.

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[31] See [GLC23].

[32] See [GOB23,Ngu23].

[33] What on earth's that? See David Gross's papers.

Summary

We've introduce RT and outline a number of applications.

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What you can do with RT:

1. Find new transforms
2. Find applications of existing transforms
3. Compute moments of random group actions

Hope this helps for some of the talks during the week!

Thank You!

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Advertisement Warning: **Soon open applications for our Quantum Computing Summer School at Los Alamos** (10 weeks, from June to August 2025)

Link not there yet but will be soon! We will post

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Marco Cerezo
@MvsCerezo

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