Computing exact moments of local random quantum circuits via tensor networks[a]

Paolo Braccia, Pablo Bermejo, Lukasz Cincio, and Marco Cerezo

The following extended abstract is based on [1].

Introduction: Computing the moments $\mathbb{E}_U[\mathrm{Tr}\left[U\rho U^\dagger O\right]^t]$ of expectation values measured at the output of random quantum circuits has become a task of paramount importance in quantum information science. For instance, their analysis can help determine conditions leading to non-classical simulability and exponential quantum advantage [2–7], the onset of quantum chaos [8–10], and the presence of local minima and barren plateaus in variational quantum computing [11–22]. If we assume that the circuit U is composed of local Haar random gates [9, 10, 23–27], one can map the problem of computing the moments to a Markov chain-like process, which enables the use of tools from classical statistical mechanics [10, 14, 18, 28–31]. While very successful at producing upper and lower bounds for the moments, these methods usually require special assumptions on the type of gates as well as on their arrangement. Furthermore, if one uses Monte Carlo (MC) sampling techniques to exactly computing the moments [14], one could require a prohibitively large number of samples due to MC's additive error, or suffer from the sign problem [32, 33]. In this work we propose *using Tensor Networks (TN) to compute the moments of the random quantum circuit's expectation values.* At its core, our work leverages the vectorization picture to represent the circuit's initial state and the measurement operator t-th fold products as Matrix Product State (MPS) [34, 35] and the local moment matrices as local gates in a TN (see Fig. 1).

Framework: In what follows we will consider a random unitary quantum circuit U acting on an n-qudit Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. We further assume that the circuit takes the form $U = \prod_{l=1}^{L} U_{\gamma_l}$, i.e., that it is composed of L local gates acting on $k_l \leqslant k$ qudits according to some topology $\mathcal{T} = \{\gamma_l\}_{l=1}^{L}$. Here, each $\gamma_l \in [n]^{\otimes k_l}$ is a tuple of (non-repeating) k_l indexes ranging between 1 and n which determine the set of qubits that U_{γ_l} acts on. For instance, if the circuit is a one-dimensional ansatz acting on alternating pairs of qudits (see Fig. 1), then we would have $\mathcal{T} = \{(1,2),(3,4),\dots(n-1,n),(2,3),\dots\}$. Moreover, we will assume that each U_{γ_l} belongs to some unitary Lie group $G_l \subseteq U(d^k)$. The t-th moment of the expectation value is

$$\mathbb{E}_U[(\mathrm{Tr}\Big[\rho U^\dagger O U\Big])^t] = \mathrm{Tr}\Big[\mathbb{E}_U[U^{\otimes t}\rho^{\otimes t}(U^\dagger)^{\otimes t}]O^{\otimes t}\Big] = \langle\langle \rho^{\otimes t}|\widehat{\tau}^{(t)}|O^{\otimes t}\rangle\rangle\,. \tag{1}$$

Here, the expectation value is taken over the set of unitaries obtained by sampling each local U_{γ_l} independently and identically distributed (i.i.d.) according to the Haar measure $d\mu_l$ over G_l. As such, we can express $\mathbb{E}_U = \mathbb{E}_{G_L}\cdots\mathbb{E}_{G_1}$, with $\mathbb{E}_G[f(U)] = \int_G d\mu(U) f(U)$ the Haar integral over the group G. In the last equality of Eq. (1), we have expressed the moment in the vectorization formalism, and we have defined

$$\widehat{\tau}^{(t)} = \int d\mu(U) U^{\otimes t}\otimes (U^*)^{\otimes t} = \prod_{l=1}^{L}\left(\int_{G_l} d\mu(U_{\gamma_l}) U_{\gamma_l}^{\otimes t}\otimes (U_{\gamma_l}^*)^{\otimes t}\right) = \prod_{l=1}^{L}\widehat{\tau}_{G_l}^{(t)}\,. \tag{2}$$

Above, $\widehat{\tau}^{(t)}$ is the t-th moment operator for the whole circuit, while $\widehat{\tau}_l^{(t)}$ are the t-th moments operators of the each of the local gates. Importantly, given that each $\widehat{\tau}_{G_l}^{(t)}$ is a projector into its local t-th order commutant, we can interpret this object as a process matrix in the bases arising from the local commutants of neighboring gates. In turn, such representation will map $\widehat{\tau}^{(t)}$ to a sequence of L such process matrices arranged according

[a] Report Number: LA-UR-24-21570

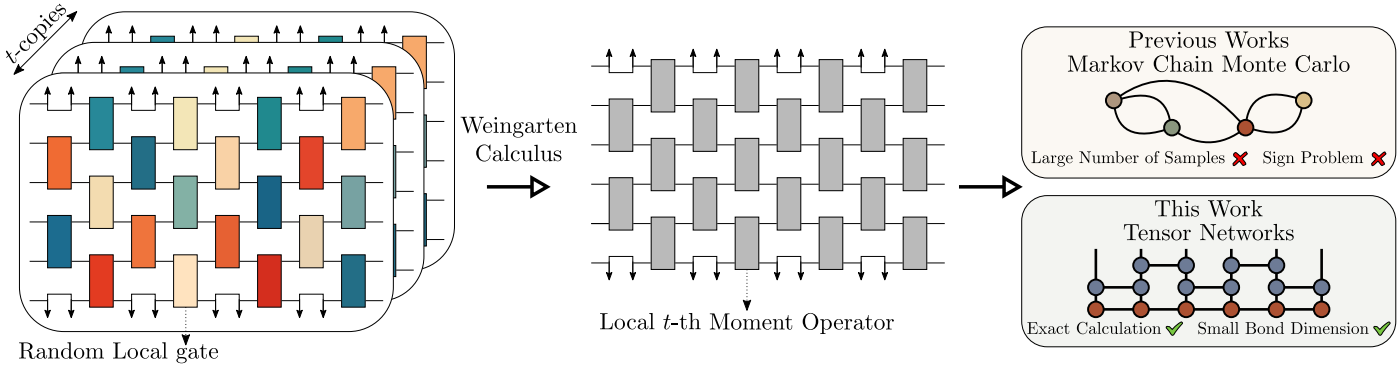

Figure 1: Schematic representation of our main results. Weingarten calculus allows to map $\mathbb{E}_U[\mathrm{Tr}\left[\rho U^\dagger O U\right]^t]$ to computing the vector inner product $\langle\langle \rho^{\otimes t}|\widehat{\tau}^{(t)}|O^{\otimes t}\rangle\rangle$, where $|\rho^{\otimes t}\rangle\rangle$ and $|O^{\otimes t}\rangle\rangle$ are respectively the vectorized t-th fold tensor product of the initial state and measurement, and where $\widehat{\tau}^{(t)}$ is the product of the local gates t-th fold moment operators. Previous works interpret $\widehat{\tau}^{(t)}$ as a Markov chain-like process and use MC sampling techniques to compute the inner product. We instead evaluate it by expressing $|\rho^{\otimes t}\rangle\rangle$ and $|O^{\otimes t}\rangle\rangle$ as MPSs, and $\widehat{\tau}^{(t)}$ as a TN with local gates.

to the topology of $\mathcal{T}$. This simple, albeit extremely important, interpretation allows us to think of Eq. (1) as the inner product between the vector $|O^{\otimes 2}\rangle\rangle$ and $|\rho^{\otimes 2}\rangle\rangle$ which are evolved through a sequence of local gates.

Main Results: Given the issues that can arise when treating the Markov chain-like problem with MC, we ask the question: *Can we use tensor networks, instead of Monte Carlo sampling, to exactly compute the moments of expectation values obtained from quantum circuits composed of local random gates?* In this work we show that not only the answer is *yes*, but that this approach has several advantage over existing MC-based methods. Our main contributions are:

- Establishing a general TN formalism to compute exact moments of random quantum circuits, whichever their topology, locality of the gates, and the local groups the gates are sampled from. At its core, our method is based on expressing $|\rho^{\otimes t}\rangle\rangle$ and $|O^{\otimes t}\rangle\rangle$ as MPSs, and $\widehat{\tau}^{(t)}$ as a TN with local gates.

- Using representation theory to deriving theoretical results which analyze the local dimension of the tensor, as well as presenting bounds for the maximum bond dimension of the matrix product states that deep circuits can produce.

- Showcasing the effectiveness of our method with extremely large scale numerical studies. We focus on $t = 2$ moments, that are crucial to diagnose the emergence of the Barren Plateaus or the onset of output probabilities anticoncentration phenomenon [36]. We compare our methods against MC sampling, revealing that we can indeed compute the moments with much higher precision when using TN (see Fig. 2), and also be able to tackle tasks where Monte Carlo would exhibit sign problems. Furthermore, we illustrate that our methods can efficiently compute the moments for circuits **acting on thousands of qubits, and composed of thousands of gates** as shown in Fig. 3. Finally, we use our TN tools to present numerical evidence that circuits composed of local random *orthogonal* gates anticoncentrate at logarithmic depth [37].

Implications and Future Directions: In this work we present a novel tool for computing moments of random quantum circuits composed of local gates. Our approach not only allow us to evaluate quantities that would be otherwise intractable, but it also enables new and exciting research directions. For instance, we note

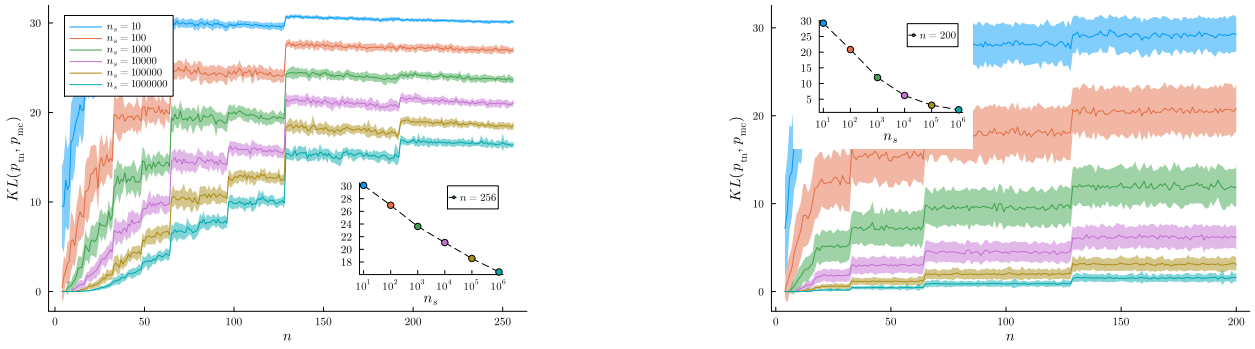

Figure 2: We compute and compare a probability distribution of moments via TNs (p_{tn}) and MC sampling (p_{mc}) via their Kullback-Leibler (KL) divergence. Results show that even at $n_s = 10^6$ MC samples, TN always performs better than MC. The inset shows that a supra-exponential increase in MC samples is needed to decrease the KL by a constant value. Left panel corresponds to a Quantum Convolutional Neural Network [38] (QCNN), while the right to a one-dimensional Hardware Efficient Ansatz (HEA). In both cases all two-qubit gates are sampled i.i.d. from $G = U(4)$

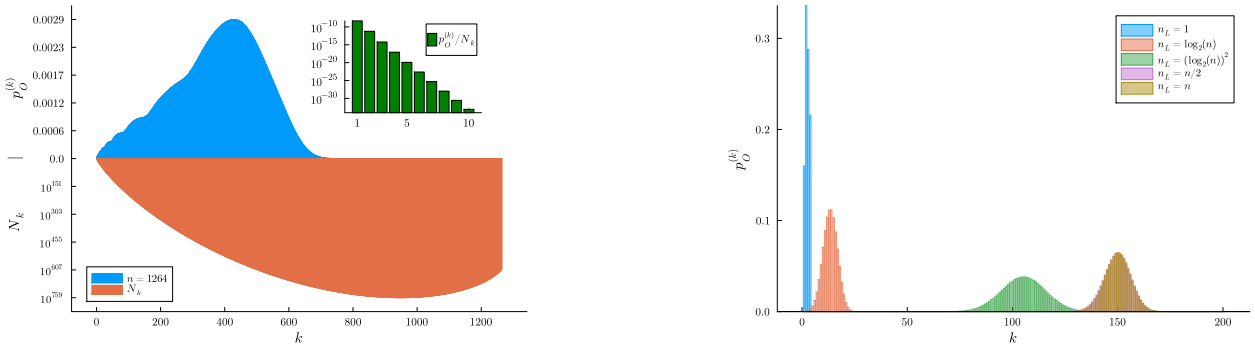

Figure 3: Left: Second moment probability distribution for a very large QCNN with gates sampled i.i.d. from $G = U(4)$ acting on $n = 1264$ qubits and having thousands of gates (blue). Note that full density matrix simulation of these expectation values would be beyond any plausible supercomputer as it would require over 10^{1632} bits to save each amplitude to machine precision. Right: Probability distribution for an HEA with gates sampled from $G = U(4)$ acting on $n = 200$ qubits as a function of the number of layers n_L. We note that the distributions for $n_L = n/2$ and $n_L = n$ are completely indistinguishable, signaling the convergence of the circuit to being a design over $U(2^n)$.

that the representation of the circuit moment operator itself could be used to learn properties of the quantum circuit independently of the initial state and measurement operator. For instance, we could use density matrix renormalization group techniques to obtain its eigenvalues, and thus be able to predict the number of layers needed for the circuit to become a t-design over G_U. On a similar note, it is worth highlighting that having access to a matrix product state representation of $\widehat{\tau}^{(t)}|\rho^{\otimes t}\rangle\rangle$ could enable a whole new dimension of random quantum circuit analysis, such as the study of the entanglement and entropic properties of this quantum state. Finally, we note that the proposed tensor network formalism can be readily applied to random quantum circuits with intermediate measurements, thus enabling the study of monitored random dynamics and measurement-induced criticality [39–42]. As such, given the versatility of our proposed techniques, we envision that tensor networks will quickly become a standard tool in the toolbox of quantum information scientist studying and working with circuits composed of random local gates.

[1] P. Braccia, P. Bermejo, L. Cincio, and M. Cerezo, Computing exact moments of local random quantum circuits via tensor networks, arXiv preprint arXiv:2403.01706 (2024).

[2] S. Boixo, S. V. Isakov, V. N. Smelyanskiy, R. Babbush, N. Ding, Z. Jiang, M. J. Bremner, J. M. Martinis, and H. Neven, Characterizing quantum supremacy in near-term devices, Nature Physics **14**, 595 (2018).

[3] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. S. L. Brandao, D. A. Buell, B. Burkett, Y. Chen, Z. Chen, B. Chiaro, R. Collins, W. Courtney, A. Dunsworth, E. Farhi, B. Foxen, A. Fowler, C. Gidney, M. Giustina, R. Graff, K. Guerin, S. Habegger, M. P. Harrigan, M. J. Hartmann, A. Ho, M. Hoffmann, T. Huang, T. S. Humble, S. V. Isakov, E. Jeffrey, Z. Jiang, D. Kafri, K. Kechedzhi, J. Kelly, P. V. Klimov, S. Knysh, A. Korotkov, F. Kostritsa, D. Landhuis, M. Lindmark, E. Lucero, D. Lyakh, S. Mandrà, J. R. McClean, M. McEwen, A. Megrant, X. Mi, K. Michielsen, M. Mohseni, J. Mutus, O. Naaman, M. Neeley, C. Neill, M. Y. Niu, E. Ostby, A. Petukhov, J. C. Platt, C. Quintana, E. G. Rieffel, P. Roushan, N. C. Rubin, D. Sank, K. J. Satzinger, V. Smelyanskiy, K. J. Sung, M. D. Trevithick, A. Vainsencher, B. Villalonga, T. White, Z. J. Yao, P. Yeh, A. Zalcman, H. Neven, and J. M. Martinis, Quantum supremacy using a programmable superconducting processor, Nature **574**, 505 (2019).

[4] Y. Wu, W.-S. Bao, S. Cao, F. Chen, M.-C. Chen, X. Chen, T.-H. Chung, H. Deng, Y. Du, D. Fan, *et al.*, Strong quantum computational advantage using a superconducting quantum processor, Physical Review Letters **127**, 180501 (2021).

[5] A. M. Dalzell, N. Hunter-Jones, and F. G. S. L. Brandão, Random quantum circuits anticoncentrate in log depth, PRX Quantum **3**, 010333 (2022).

[6] M. Oszmaniec, N. Dangniam, M. E. Morales, and Z. Zimborás, Fermion sampling: a robust quantum computational advantage scheme using fermionic linear optics and magic input states, PRX Quantum **3**, 020328 (2022).

[7] H.-Y. Huang, R. Kueng, G. Torlai, V. V. Albert, and J. Preskill, Provably efficient machine learning for quantum many-body problems, Science **377**, eabk3333 (2022).

[8] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Quantum entanglement growth under random unitary dynamics, Physical Review X **7**, 031016 (2017).

[9] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, otocs, and entanglement growth in systems without conservation laws, Physical Review X **8**, 021013 (2018).

[10] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Physical Review X **8**, 021014 (2018).

[11] J. R. McClean, S. Boixo, V. N. Smelyanskiy, R. Babbush, and H. Neven, Barren plateaus in quantum neural network training landscapes, Nature Communications **9**, 1 (2018).

[12] M. Cerezo, A. Sone, T. Volkoff, L. Cincio, and P. J. Coles, Cost function dependent barren plateaus in shallow parametrized quantum circuits, Nature Communications **12**, 1 (2021).

[13] A. Pesah, M. Cerezo, S. Wang, T. Volkoff, A. T. Sornborger, and P. J. Coles, Absence of barren plateaus in quantum convolutional neural networks, Physical Review X **11**, 041011 (2021).

[14] J. Napp, Quantifying the barren plateau phenomenon for a model of unstructured variational ansätze, arXiv preprint arXiv:2203.06174 (2022).

[15] M. Ragone, B. N. Bakalov, F. Sauvage, A. F. Kemper, C. O. Marrero, M. Larocca, and M. Cerezo, A unified theory of barren plateaus for deep parametrized quantum circuits, arXiv preprint arXiv:2309.09342 (2023).

[16] E. Fontana, D. Herman, S. Chakrabarti, N. Kumar, R. Yalovetzky, J. Heredge, S. Hari Sureshbabu, and M. Pistoia, The adjoint is all you need: Characterizing barren plateaus in quantum ansätze, arXiv preprint arXiv:2309.07902 (2023).

[17] N. L. Diaz, D. García-Martín, S. Kazi, M. Larocca, and M. Cerezo, Showcasing a barren plateau theory beyond the dynamical lie algebra, arXiv preprint arXiv:2310.11505 (2023).

[18] A. Letcher, S. Woerner, and C. Zoufal, Tight and efficient gradient bounds for parameterized quantum circuits, arXiv preprint arXiv:2309.12681 (2023).

[19] S. Thanaslip, S. Wang, N. A. Nghiem, P. J. Coles, and M. Cerezo, Subtleties in the trainability of quantum machine learning models, Quantum Machine Intelligence **5**, 21 (2023).

[20] E. R. Anschuetz and B. T. Kiani, Quantum variational algorithms are swamped with traps, Nature Communications **13**, 7760 (2022).

[21] E. R. Anschuetz, Critical points in quantum generative models, International Conference on Learning Representations (2022).

[22] L. Monbroussou, J. Landman, A. B. Grilo, R. Kukla, and E. Kashefi, Trainability and expressivity of hamming-weight preserving quantum circuits for machine learning, arXiv preprint arXiv:2309.15547 (2023).

[23] P. Hayden and J. Preskill, Black holes as mirrors: quantum information in random subsystems, Journal of High Energy Physics **9**, 120 (2007).

[24] Y. Sekino and L. Susskind, Fast scramblers, Journal of High Energy Physics **2008**, 065 (2008).

[25] W. Brown and O. Fawzi, Scrambling speed of random quantum circuits, arXiv preprint arXiv:1210.6644 (2012).

[26] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, Towards the fast scrambling conjecture, Journal of High Energy Physics **2013**, 1 (2013).

[27] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, Chaos in quantum channels, Journal of High Energy Physics **2016**, 1 (2016).

[28] N. Hunter-Jones, Unitary designs from statistical mechanics in random quantum circuits, arXiv preprint arXiv:1905.12053 (2019).

[29] B. Barak, C.-N. Chou, and X. Gao, Spoofing linear cross-entropy benchmarking in shallow quantum circuits, arXiv preprint arXiv:2005.02421 (2020).

[30] A. W. Harrow and S. Mehraban, Approximate unitary t-designs by short random quantum circuits using nearest-neighbor and long-range gates, Communications in Mathematical Physics **401**, 1531 (2023).

[31] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, Holographic duality from random tensor networks, Journal of High Energy Physics **2016**, 1 (2016).

[32] G. Pan and Z. Y. Meng, Sign problem in quantum monte carlo simulation, in *Encyclopedia of Condensed Matter Physics (Second Edition)* (Academic Press, 2024) second edition ed., pp. 879–893.

[33] X. Mi, P. Roushan, C. Quintana, S. Mandra, J. Marshall, C. Neill, F. Arute, K. Arya, J. Atalaya, R. Babbush, *et al.*, Information scrambling in quantum circuits, Science **374**, 1479 (2021).

[34] R. Orús, A practical introduction to tensor networks: Matrix product states and projected entangled pair states, Annals of Physics **349**, 117 (2014).

[35] J. Biamonte and V. Bergholm, Tensor networks in a nutshell, arXiv preprint arXiv:1708.00006 (2017).

[36] D. Hangleiter, J. Bermejo-Vega, M. Schwarz, and J. Eisert, Anticoncentration theorems for schemes showing a quantum speedup, Quantum **2**, 65 (2018).

[37] A. M. Dalzell, N. Hunter-Jones, and F. G. S. L. Brandão, Random quantum circuits transform local noise into global white noise, arXiv preprint arXiv:2111.14907 (2021).

[38] I. Cong, S. Choi, and M. D. Lukin, Quantum convolutional neural networks, Nature Physics **15**, 1273 (2019).

[39] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. Ludwig, Measurement-induced criticality in random quantum circuits, Physical Review B **101**, 104302 (2020).

[40] Y. Bao, S. Choi, and E. Altman, Theory of the phase transition in random unitary circuits with measurements, Physical Review B **101**, 104301 (2020).

[41] A. Nahum, S. Roy, B. Skinner, and J. Ruhman, Measurement and entanglement phase transitions in all-to-all quantum circuits, on quantum trees, and in landau-ginsburg theory, PRX Quantum **2**, 010352 (2021).

[42] M. P. Fisher, V. Khemani, A. Nahum, and S. Vijay, Random quantum circuits, Annual Review of Condensed Matter Physics **14**, 335 (2023).