Architectures and random properties of symplectic quantum circuits

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Parametrized and random unitary (or orthogonal) n-qubit circuits play a central role in quantum information. As such, one could naturally assume that circuits implementing symplectic transformation would attract similar attention. However, this is not the case, as $\mathbb{SP}(d/2)$—the group of $d \times d$ unitary symplectic matrices—has thus far been overlooked. In this work, we aim at starting to right this wrong. We begin by presenting a universal set of generators $\mathcal{G}$ for the symplectic algebra $i\mathfrak{sp}(d/2)$, consisting of one- and two-qubit Pauli operators acting on neighboring sites in a one-dimensional lattice. Here, we uncover two critical differences between such set, and equivalent ones for unitary and orthogonal circuits. Namely, we find that the operators in $\mathcal{G}$ cannot generate arbitrary local symplectic unitaries and that they are not translationally invariant. We then review the Schur-Weyl duality between the symplectic group and the Brauer algebra, and use tools from Weingarten calculus to prove that Pauli measurements at the output of Haar random symplectic circuits can converge to Gaussian processes. As a by-product, such analysis provides us with concentration bounds for Pauli measurements in circuits that form t-designs over $\mathbb{SP}(d/2)$. To finish, we present tensor-network tools to analyze shallow random symplectic circuits, and we use these to numerically show that computational-basis measurements anti-concentrate at logarithmic depth.

INTRODUCTION

Parametrized and random unitary quantum circuits play a central role in quantum computing and quantum information, and consequently they have been the subject of numerous studies [1–13]. Certain families of quantum circuits belonging to subgroups of the unitary group have also been studied in detail, such as circuits composed of Clifford gates [14, 15] or matchgates [16–21]. However, other subgroups have received much less attention. In particular, the compact symplectic group has been mostly neglected in the recent literature, despite its importance in random matrix theory [22] and classical [23] and quantum [24] dynamics. In this work, we rigorously address the study of quantum circuits that implement symplectic unitary transformations, discovering easy-to-implement architectures and computing many of their random properties (both in the deep and shallow regimes), see Fig. 1.

BACKGROUND

We recall that the standard representation of the compact symplectic group $\mathbb{SP}(d/2) := \mathbb{SP}(d;\mathbb{C}) \cap \mathbb{SU}(d)$ consists of all $d \times d$ unitary matrices (with d an even number), such that any $S \in \mathbb{SP}(d/2)$ satisfies the relation

$$S^T \Omega S = \Omega \,, \tag{1}$$

where Ω is a non-degenerate anti-symmetric bilinear form. In other words, $\mathbb{SP}(d/2)$ is the group of unitary matrices that preserve the product $\boldsymbol{x}^T \Omega\, \boldsymbol{y}$ for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^d$. Its associated Lie algebra is denoted $\mathfrak{sp}(d/2)$. Here, we remark that Ω in Eq. (1) is not uniquely defined. Typically, one uses the Darboux basis—or canonical form– in which Ω takes the form

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_{d/2} \\ -\mathbb{1}_{d/2} & 0 \end{pmatrix} , \tag{2}$$

with $\mathbb{1}_{d/2}$ being the $d/2 \times d/2$ identity matrix. In this work we will assume that Ω is given by Eq. (2), and we will focus on the case when $d = 2^n$ so that the symplectic unitaries act on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ of n qubits. With this choice one can verify that

$$\Omega = iY \otimes \mathbb{1}^{\otimes n-1} \,, \tag{3}$$

with Y the Pauli matrix and $\mathbb{1}$ the 2×2 identity.

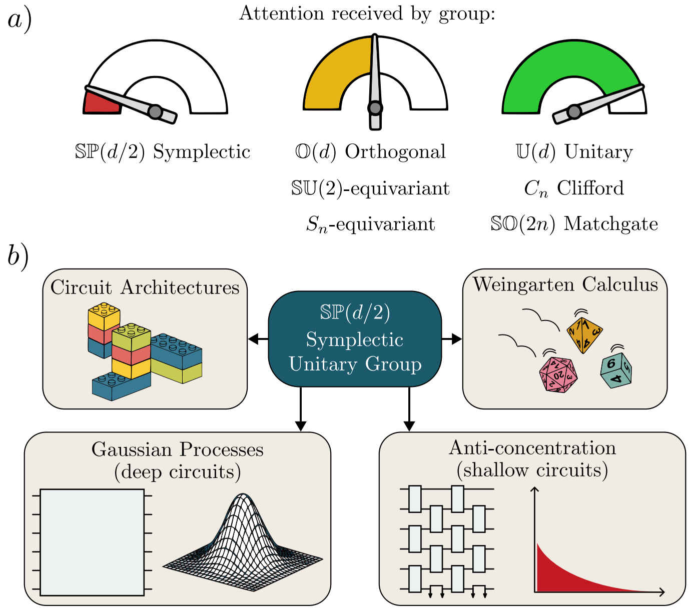

Figure 1. **Schematic representation of our main results**. a) When compared against other groups, the compact group $\mathbb{SP}(d/2)$ of $d \times d$ symplectic unitaries has received considerably less attention. b) Here we introduce tools to study $\mathbb{SP}(d/2)$, such as presenting easy-to-implement circuit architectures capable of producing any symplectic evolution. We also review the Weingarten calculus for this group and use it to study properties of random symplectic circuits, like their convergence to Gaussian processes (deep circuits) or the emergence of anti-concentration (shallow circuits).

MAIN RESULTS

Architectures for symplectic circuits

Our first contribution is to show that by taking the canonical Ω as in Eq. (3), we can find a set local generators $\mathcal{G}$ for which circuits of the form

$$U = \prod_l e^{i\theta_l H_l} \,, \tag{4}$$

where θ_l are real-valued parameters and $H_l \in \mathcal{G}$, are universal and can therefore produce any unitary in $\mathbb{SP}(d/2)$. In particular, we prove the following theorem.

Theorem 1. *The set of unitaries of the form in Eq. (4), with generators taken from*

$$\mathcal{G} = \{Y_i\}_{i=1}^{n} \cup \{X_i Y_{i+1}, Y_i X_{i+1}\}_{i=2}^{n-1} \cup X_1 \cup Z_1 Z_2 \,, \tag{5}$$

is universal in $\mathbb{SP}(d/2)$*, as*

$$\text{span}_{\mathbb{R}} \langle i\mathcal{G} \rangle_{\text{Lie}} = \mathfrak{sp}(d/2) \,. \tag{6}$$

Here, $\langle i\mathcal{G} \rangle_{\text{Lie}}$ *is the Lie closure of* $i\mathcal{G}$*, i.e., the set of operators obtained by the nested commutation of the elements in* $i\mathcal{G}$*.*

In Eq. (5), X_i, Y_i and Z_i denote the Pauli operators acting on the i-th qubit. The architecture presented in Theorem 1 consists of one- and two-qubit gates acting on nearest neighbors on a one-dimensional lattice, as shown in Fig. 2, which renders it easy to implement on near-term hardware.

It is interesting to comment on two important differences between symplectic circuits and those implementing transformations from the unitary or orthogonal groups. First, the fact that symplectic circuits cannot be generated from translationally-invariant local operators (i.e., which are the same on each pair of adjacent qubits). Indeed, we have shown that the structure of the symplectic Lie algebra and its associated Lie group places a privileged role on a single qubit in the system, thus breaking typical qubit-exchange symmetries appearing when working with $\mathbb{U}(2^n)$ or $\mathbb{O}(2^n)$. And second, the fact that they cannot implement arbitrary locally-symplectic transformations, as circuits composed of locally-symplectic gates are able to produce non-symplectic unitaries.

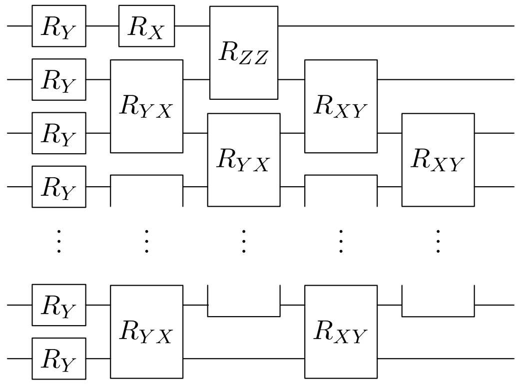

Figure 2. **Quantum circuits for symplectic unitaries**. Example of the basic building block for the implementation of symplectic unitary transformations on a quantum computer. The notation R_{H_l} stands for $e^{i\theta_l H_l}$, with independent θ_l angles in each gate. As stated in Theorem 1, the Lie closure of the generators appearing in this circuit, which are not translationally invariant, produces $\mathfrak{sp}(d/2)$. This implies that any symplectic unitary from the $\mathbb{SP}(d/2)$ group can be implemented by a quantum circuit architecture consisting of blocks of this form.

Random properties of symplectic circuits

We also study random properties of symplectic quantum circuits via Weingarten calculus. Namely, we study the convergence of the circuits' outputs to Gaussian Processes (GPs) in three different regimes (Theorems 2,3,4 in our paper), and the emergence of anti-concentration (Theorem 5).

For the GPs analysis, we consider a setting where we are given a set $\mathscr{D} = \{\rho_1, \dots, \rho_m\}$ of real-valued n-qubit quantum states on a d-dimensional Hilbert space (i.e., $\rho_j = \rho_j^T \; \forall j$). We then take the m states from $\mathscr{D}$ and send them through a unitary U which is sampled according to the Haar measure over $\mathbb{SP}(d/2)$. At the output of the circuit we measure the expectation value of a Pauli operator O taken from $i\mathfrak{sp}(d/2)$. This leads to a set of quantities of the form $C(\rho_j) = \mathrm{Tr}[U\rho_j U^\dagger O]$, which we collect in a length-m vector

$$\mathscr{C} = (C(\rho_1), \dots, C(\rho_m)) \,. \tag{7}$$

We will say that $\mathscr{C}$ forms a GP iff it follows a multivariate Gaussian. For instance, we can prove the following theorem.

Theorem 2. *Let $\mathscr{C}$ be a vector of expectation values of the Hermitian operator O over a set of states from $\mathscr{D}$, as in Eq. (7). If $\mathrm{Tr}[\rho_j\rho_{j'}] \in \Omega\left(1/\operatorname{poly}(\log(d))\right)$ and $\left|\mathrm{Tr}[\Omega\rho_j\Omega\rho_{j'}]\right| \in o\left(1/\operatorname{poly}(\log(d))\right) \; \forall j, j'$, then in the large d-limit $\mathscr{C}$ forms a GP with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix*

$$\boldsymbol{\Sigma}_{j,j'} = \frac{\mathrm{Tr}[\rho_j\rho_{j'}]}{d} \,. \tag{8}$$

Regarding anti-concentration [25, 26] (which is a key ingredient in quantum supremacy experiments [27–33]), we analytically prove that computational-basis measurements in circuit ensembles that form 2-designs over $\mathbb{SP}(d/2)$ will anti-concentrate. Furthermore, we use tensor-network tools developed in Ref. [34] to numerically show that anti-concentration will appear at logarithmic depth in locally-random symplectic circuits.

CONCLUSIONS AND OUTLOOK

Our work significantly contributes to the body of knowledge of quantum circuits implementing symplectic transformations, as previous studies on the topic are nearly non-existent [35, 36]. Here, we present an easy-to-implement architecture to produce symplectic evolutions. Given the crucial role that the symplectic group plays in classical and quantum dynamical systems, our work opens up the possibility of using quantum computers to study such systems, both in the context of quantum machine learning and in the realm of fault-tolerant quantum algorithms, as symplectic unitaries can now be compiled to qubit architectures and therefore executed in currently-available quantum hardware.

Besides, the discovery that the outputs of symplectic quantum circuits can converge in distribution to GPs adds to the literature showing convergence of quantum neural networks to GPs [37–39]. This type of connection was crucial in classical machine learning in order to develop a theoretical understanding and a convergence theory for artificial neural networks, and so it is expected to be in the field of quantum machine learning.

LINK TO THE MANUSCRIPT

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