Stabilizer Tensor Networks: universal quantum simulator on a basis of stabilizer states

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Simulation of quantum computing is currently driving science in fields like condensed matter physics [1, 2] or quantum chemistry [3–5] while we don't have large, error-corrected devices, and it is also useful to test quantum advantage claims [6–8] made by cutting-edge devices [9, 10]. However, simulating efficiently beyond a few dozen qubits requires characterising the target system properly. Thus, a large effort is put towards identifying which states are easy and why, in the absence of a universal description of simulablity. A useful tool for this task are resource theories [11], which characterize the operations that are easy to do (free operations) in a certain framework.

In this article we focus our interest on entanglement [12] and stabilizer rank [13, 14]. The interest in relating different resources, particularly these two, is not new. Previous research has found some of these states present maximal entanglement, at least in the bipartite sense [15], although most types of entangled states are not achievable with these circuits. Recently, magic in MPS states has also been characterized and looked into [16], and it is noteworthy that separable states with a lot of magic are complex in the stabilizer formalism, even though they are trivial to simulate with resource theories of entanglement. This means that these resources are in some sense orthogonal, as depicted in Fig. 1a. We propose a new formalism that unifies these two resources and can be used for simulating arbitrary circuits.

Entanglement as a resource for simulation is usually characterized with bipartite entanglement between sectors of the system. This is the case of circuit cutting [18], entanglement forging [19] or tensor networks [20]. These simulations rely on limited entanglement, mostly between close neighbours [21], or on a hierarchical structure of entanglement [22, 23]. Free operations are single-qubit (local) gates and classical communication [24, 25]. For systems with limited entanglement, it is advantageous to use tensor networks. We focus on an MPS structure (see Fig. 1b2), a 1D tensor network to encode the amplitudes of a quantum state:

$$\mathcal{T}^{i_1 i_2 \dots i_n} = \sum_{k_1 k_2 \dots k_{n-1}} (T_1)^{i_1}_{k_1} (T_2)^{i_2 k_1}_{k_2} (T_3)^{i_3 k_2}_{k_3} \dots (T_N)^{i_n k_{n-1}} \tag{1}$$

In this structure, the dimension χ of a given bond corresponds to the *entanglement* between the two subsystems it connects. On the other hand, the stabilizer tableau formalism [26] can simulate any circuit that consists only of Clifford gates efficiently with a classical computer. The formalism is based on the generators of a stabilizer group $\mathcal{S}$ because those define a unique state $|\psi_{\mathcal{S}}\rangle$. These generators can be decomposed into some boolean vectors and stored in a tableau (see Fig. 1b1), then this tableau can be updated efficiently under the action of any Clifford gate. An arbitrary state $|\psi\rangle$ might be decomposed into a superposition of ξ such stabilier states, which is related to its *non-stabilizerness.*

In the article, we use the stabilizer formalism as a basis for the Hilbert space (and prove it works in lemma 1) as was proposed in [27], then we use an MPS to encode the amplitudes of this decomposition as $|\nu\rangle = \sum_i \nu_i |i\rangle$. It is already known how Clifford gates change the basis [17], but we propose two lemmas that establish how non-Clifford gates and measurements affect $|\nu\rangle$. We summarize all of these into some update rules that are then used in our python implementation to enable universal simulation. The key ingredient to stabilizer tensor networks is allowing the basis to change. The tableau algorithm replaces the computational basis with a basis of stabilizer states, which are able to store some entanglement. In some sense, entanglement moves from the tensor network $|\nu\rangle$ representation into the basis, at the price of single qubit gates possibly becoming entangling on the amplitudes of $|\nu\rangle$. In general, this only happens if we already had entangling gates in the circuit and thus the complete unitary was entangling in the first place, so we argue that the formalism is not generating fictitious entanglement. Instead, we say we store *potential entanglement* in the basis.

We propose two examples, low entanglement and low stabilizer rank, to showcase how the formalism can be advantageous for both resources. In the original tableau formalism [17], it was already described how we can simulate any

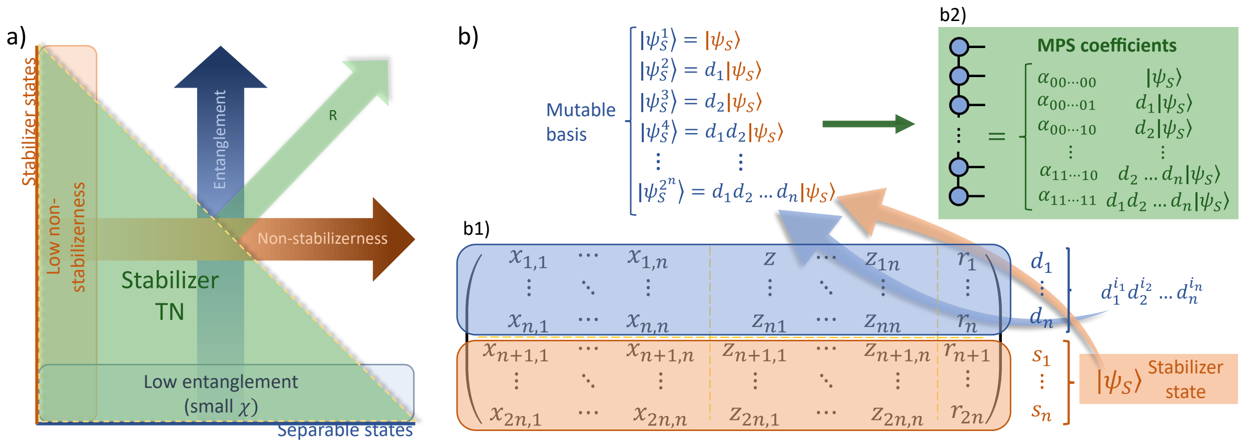

FIG. 1. Summary of our formalism. In a), we classify states under two different resources. For entanglement, in blue, (non-stabilizerness, orange), axis y=0 (x=0) represents the free states, while its adjacent region represents states with low amounts of entanglement (non-stabilizerness), which are classically simulatable with tensor networks (stabilizer tableaus). Stabilizer tensor networks, in green, can simulate the free states of both, relating to a different resource R. In b) we show how stabilizer TN joins the other methods: the tableau formalism (b1) encodes a stabilizer state and a set of destabilizer generators which are used to form a basis for the Hilbert space, following the tableau formalism [17]. The amplitudes of decomposing a state $|\psi\rangle$ into the changing basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ are stored in a tensor network (green, b2).

circuit and encode any state in the n qubit Hilbert space with a superposition of tableaus, which grows exponentially in the amount of non-Clifford gates t. Instead, our formalism takes advantage of the TN; we show with state $|T\rangle^n$:

$$|T\rangle^n = \prod_{i=1}^{n} T_i \prod_{i=1}^{n} H_i \, |0\rangle^{\otimes n} \,, \tag{2}$$

that it can be prepared only with free operations and is trivial to represent on an MPS with $\chi = 1$ for any n. On the other hand, some stabilizer states have been shown to have maximum bipartite entanglement [15]; preparing these states with a Clifford circuit meants that they are an element of the basis $\mathcal{B}(\mathcal{S},\mathcal{D})$ in a stabilizer tensor network, so they are also trivial to represent with $\xi = \tilde{\xi} = 1$ despite being expensive with a regular MPS.

The cases where there are *low* amounts of these resources are also discuss. We are able to bind how much the entanglement can grow in our TN for a non-Clifford update due to the *potential entanglement* in the basis, finding a worst case scenario of $\chi' = 2^4\chi$ when using an MPS (which can be improved with other topologies). We also perform simulations with uniformly random Clifford gates to see that on average it is much lower ($\sim 2^{2.46}$) and that it does not grow with $n \to \infty$. Our results ensure efficient simulation with a low amount of T-gates, matching in the worst case the complexity of current non-Clifford tableau simulations. In addition to these examples, there is likely to be a different resource R that captures the power of the approach and defines, with a single metric, whether a state can be efficiently represented with it or not (Fig. 1a), and we show that it must be non-trivially related to these two resources.

The development of this formalism is linked to recent research directions for quantum simulation. Firstly, stabilizer tensor networks can be directly transformed into a Clifford Enhanced Matrix Product State, recently formalised [28], which also ties in relevant techniques in simulation from the last years [29]. This area can greatly benefit from studying the optimality of our simulation approach through different gate decompositions and circuit compilations. The resource R, on the other hand, is obviously linked to phase separations found in the Hilbert space landscape when studying magic and entanglement jointly [30]. Also, bounding χ of the MPS and checking the accuracy of results on states with different amounts of entanglement, magic, or other resources is a strong candidate to characterize the resource R, possibly finding new families of states that can be easily simulated and connecting with magic in MPS research. In fact, being able to relate χ to a magnitude other than bipartite entanglement opens up the field of tensor networks to the use of other resources that we know how to simulate, and the changing basis in our technique offers a formal basis to understand this approach. Lastly, the efficiency of our implementation can be improved (for example via the usage of the most performant python tableau simulator, STIM [31]) and integrated with new tools [32] that can facilitate the use of the technique for large scale simulations with HPC.

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