Quantum algorithms for spectral sums

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How to compute the sum of a function of the eigenvalues of a matrix

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$$A = U\Sigma U^{\dagger}$$

(positive semi-definite)

$$f : \mathbb{R} \to \mathbb{R}$$

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$$\sum_i f(\lambda_i) \quad = Tr[f(\Sigma)] = Tr[f(A)]$$

$$f : \mathbb{R} \to \mathbb{R}$$

(Can be generalized to non-square matrices using SVD)

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Why spectral sums are important?

Spectral sums are ubiquitous in many domains of science.

1 (Log-)determinant

2 von Neuman Entropy (of a graph)

3 **Schatten p-norms**

4 Trace of Inverse

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Applications in spectral graph theory

- Number of triangles
- Effective resistance between nodes
- Number of spanning trees

check the ~150 citations in our paper to explore other applications!

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How to compute spectral sums on a quantum computer?

https://arxiv.org/abs/2011.06475

Toolkit

A plug-and-play framework for computing $Tr[f(A)]$

01

Block-encoding of A

- sparse access
- QRAM
- state preparation
- …

Classical data and quantum computers?

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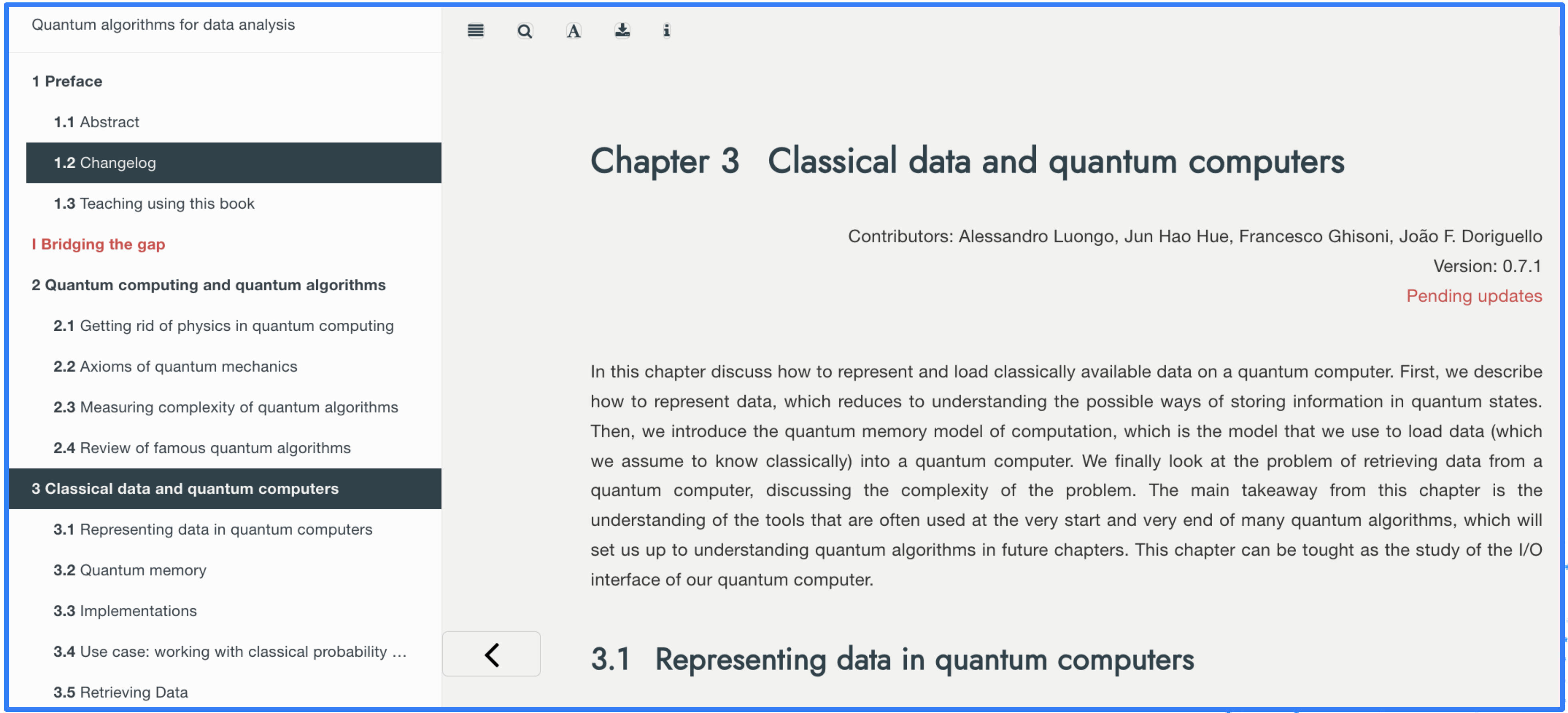

Quantum algorithms for data analysis

Chapter 3 Classical data and quantum computers

Contributors: Alessandro Luongo, Jun Hao Hue, Francesco Ghisoni, João F. Doriguello

Version: 0.7.1

Pending updates

In this chapter discuss how to represent and load classically available data on a quantum computer. First, we describe how to represent data, which reduces to understanding the possible ways of storing information in quantum states. Then, we introduce the quantum memory model of computation, which is the model that we use to load data (which we assume to know classically) into a quantum computer. We finally look at the problem of retrieving data from a quantum computer, discussing the complexity of the problem. The main takeaway from this chapter is the understanding of the tools that are often used at the very start and very end of many quantum algorithms, which will set us up to understanding quantum algorithms in future chapters. This chapter can be tought as the study of the I/O interface of our quantum computer.

3.1 Representing data in quantum computers

Classical data and quantum computers?

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- $1/x$
- x^p
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Or your favorite way of computing $f(A)$

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Trace estimation algos

- Relative error
- Absolute error
- Trace of $Tr[A^T A]$

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Trace estimation algos

- Relative error
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Quantum linear algebra is all you need for Spectral Sums

Other tools and new tricks

- Compute trace of $Tr[A^T A]$ using $\tilde{O}\left(\frac{\alpha\sqrt{n}}{\epsilon\sqrt{\mathrm{Tr}[B^\dagger B]}}\right)$ operations [1]

- How to go from absolute error ($|a - E| < \epsilon$)
  to relative error. ($|a - E| < \epsilon a$).
  - Problems solved many times in CS (e.g. [2]) … but can you find a nice plug-and-play theorem? Now you can! (our paper)

- We generalized for k>2 the algorithm for computing block-encoding of A^k . Cost: $O(k\alpha)$ instead of $O(\alpha^k)$

[1] Van Apeldoorn, J., Gilyen, A., Gribling, S., Wolf, R.: Quantum sdp-solvers: Better upper and lower bounds. Quantum
[2] Chowdhury, A.N., Somma, R.D., Subası, Y.: Computing partition functions in the one-clean-qubit model. Physical Review A

Algo for Shatten p-norms: $Tr[A^p]$

Theorem (Algorithm for estimating Schatten p-norms).
For $\epsilon \geq 0$, $p \in \mathbb{N}^+$, and let U_A be a (α, a)-encoding of a matrix $A \in \mathbb{R}^{m \times n}$ with $n \leq m$.

There is a quantum algorithm that computes an ϵ-relative approximation of $Tr[A^p]$ of A using U_A and $U_A^\dagger$:

- $\widetilde{O}\left(\frac{p\sqrt{n}\rho\alpha}{\|A\|\epsilon}\right)$ *times if p is even,*
- $\widetilde{O}\left(\frac{\sqrt{n}\alpha^{1.5}\rho}{\epsilon}\left(\frac{p}{2}+\kappa\right)\right)$ *times if p is odd.*

The $\widetilde{O}$ notation hides factor polylogarithmic in p, $\sqrt{n}$, α, and $1/\epsilon$.
Here, $\rho = \frac{2^{p/2}\|A\|^p}{\|A\|_p^{p/2}}$.

Idea: use $Tr[A^T A]$

Algo for Shatten p-norms: $Tr[A^p]$ (second algorithm 😎)

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Here, $\rho = \frac{2^{p/2}\|A\|^p}{\|A\|_p^{p/2}}$.

Idea: use $Tr[A^T A]$ ...
... and the polynomial approximation of monomials in [-1, 1]

Simple application: counting triangles in a graph

Previous work: [1]

$$O((\sqrt{n}\Delta(G)^{-1/6} + m^{3/4}\Delta(G)^{-1/2})\mathrm{poly}(1/\epsilon))$$

New: 2 quantum algorithms

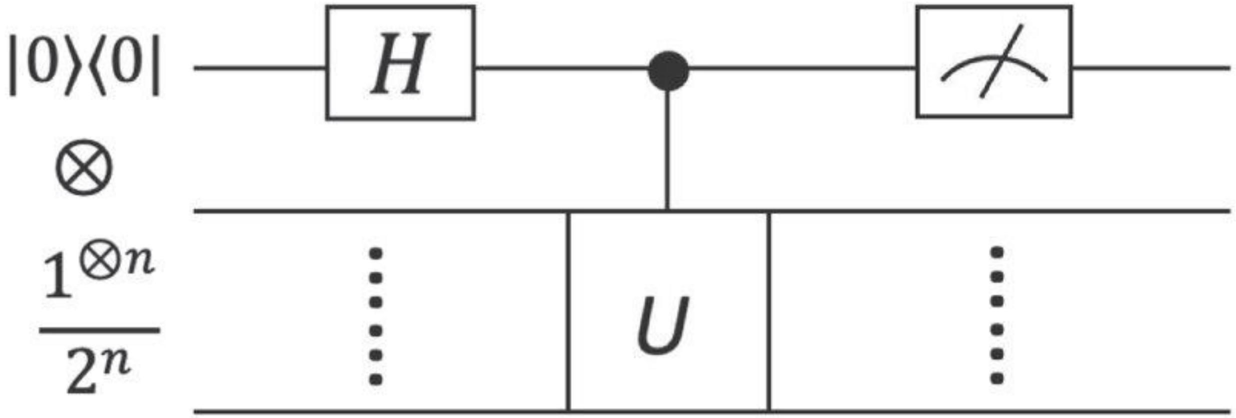
$$\tilde{O}\left(\frac{m\kappa}{\epsilon\sqrt{n\Delta(G)}}\right) \quad \tilde{O}\left(\frac{m}{\epsilon\Delta(G)}\right)$$

- n = number of nodes
- m = number of edges
- Δ(G) = number of triangles
- ε = approximation error

[1] Hamoudi, Yassine, and Frédéric Magniez. "Quantum Chebyshev's Inequality and Applications." *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*. 2019.

DQC1 and low-depth amp-est.

Deterministic one-clean-qubit model of quantum computation in 2024



DQC1: the class of (decision problems) solvable by a one-clean-qubit machine in polynomial time.

Trace estimation is a famously contained in DQC1.

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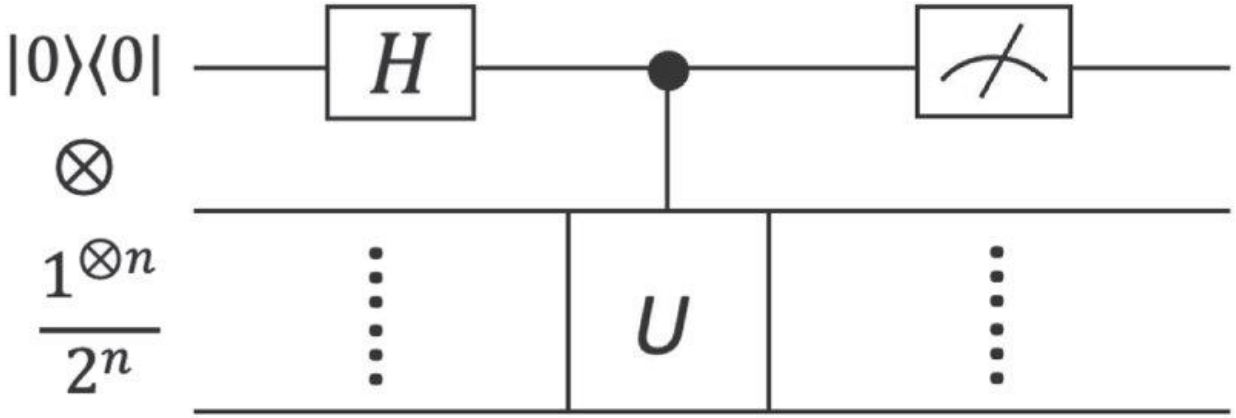

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Trace estimation is a famously contained in DQC1.

1) All the algorithms we did can be run on DQC1-computers

2) Using [1] we go from depth of $O(\frac{1}{\varepsilon})$ to a circuit depth $O\left(\frac{1}{\varepsilon^{1-\beta}}\right)$ repeated for $O(\frac{1}{\varepsilon^{1+\beta}})$ times

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Results: runtime of algorithms

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1 (Log-)determinant $\tilde{O}\left(\frac{\alpha\kappa}{\epsilon}\right)$

2 Entropy of a graph $\tilde{O}\left(\frac{n\alpha\kappa}{\epsilon s(G)}\right)$

3 **Schatten p-norms** $\tilde{O}\left(\frac{\sqrt{p}n\alpha}{\epsilon}\right)$

4 Trace of Inverse $\tilde{O}\left(\frac{\alpha^2\kappa^2}{\epsilon}\right)$

Shame on us, let's make this better!

Applications in spectral graph theory

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What to do next?

Directions in theory and practice

Tyler M-estimator
(application of log-det)

Implementations in DQC1 computers!

Low-depth circuits

Log-det of non-square matrices! 😍

New applications

Better “Trace of Inverse”

Thanks for your time.

Internship+freelance positions available: ale@{nus.edu.sg, inveriant.com}