Dual-frame optimization for informationally complete quantum measurements

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A scalable dual frame optimization protocol improves the statistical performances of estimators constructed from classical shadows and overcomplete POVMs

Operator averaging with IC-POVMs

A POVM is a set of operators $\{M_k\}$ (effects) with the following properties:

- Positivity $\forall k$: $\mathrm{Tr}[\rho M_k] \geq 0$, for any ρ
- Normalization $\sum_k M_k = \mathbb{I}$
- Born's Rule $p_k = \mathrm{Tr}[\rho M_k]$

A POVM is said to be **Informationally Complete (IC)** if it spans the space of Hermitian operators.

Expectation value estimation. Given a quantum state ρ prepared by some quantum algorithm, estimate $\langle \mathcal{O} \rangle_\rho = \mathrm{Tr}[\rho \mathcal{O}]$

Step 1. Decompose

$$\mathcal{O} = \sum_k \omega_k M_k \quad \Rightarrow \quad \langle \mathcal{O} \rangle_\rho = \sum_k \omega_k p_k$$

Step 2. Sample

$$\hat{o} = \sum_k \frac{\#k}{S} \omega_k \rightarrow \langle \mathcal{O} \rangle_\rho \quad \text{with error } \epsilon \sim \sqrt{\frac{\mathrm{Var}[\omega_k]}{S}}$$

Can be adjusted with the choice of the POVM [1], ***BUT*** *it also depends on the decomposition coefficients* ω_k. If the POVM is overcomplete, these are not unique.

Measurement dual frames

Informally, in a finite-dimensional vector space V a **frame** is a (possibly overcomplete) collection $\{v_k\} \subset V$ spanning V. For every frame, there exist at least another collection of vectors $\{\tilde{v}_k\}$, called a **dual frame**, such that for any $v \in V$ we have $v = \sum_k \langle \tilde{v}_k, v \rangle v_k = \sum_k \langle v_k, v \rangle \tilde{v}_k$.

We can think of the elements $\{\mathrm{M_k}\}$ of any IC-POVM as a **measurement frame** in the vector space of Hermitian operators of the same dimension [2]. In this case, the duals $\{\mathrm{D_k}\}$ represent estimators for quantum states:

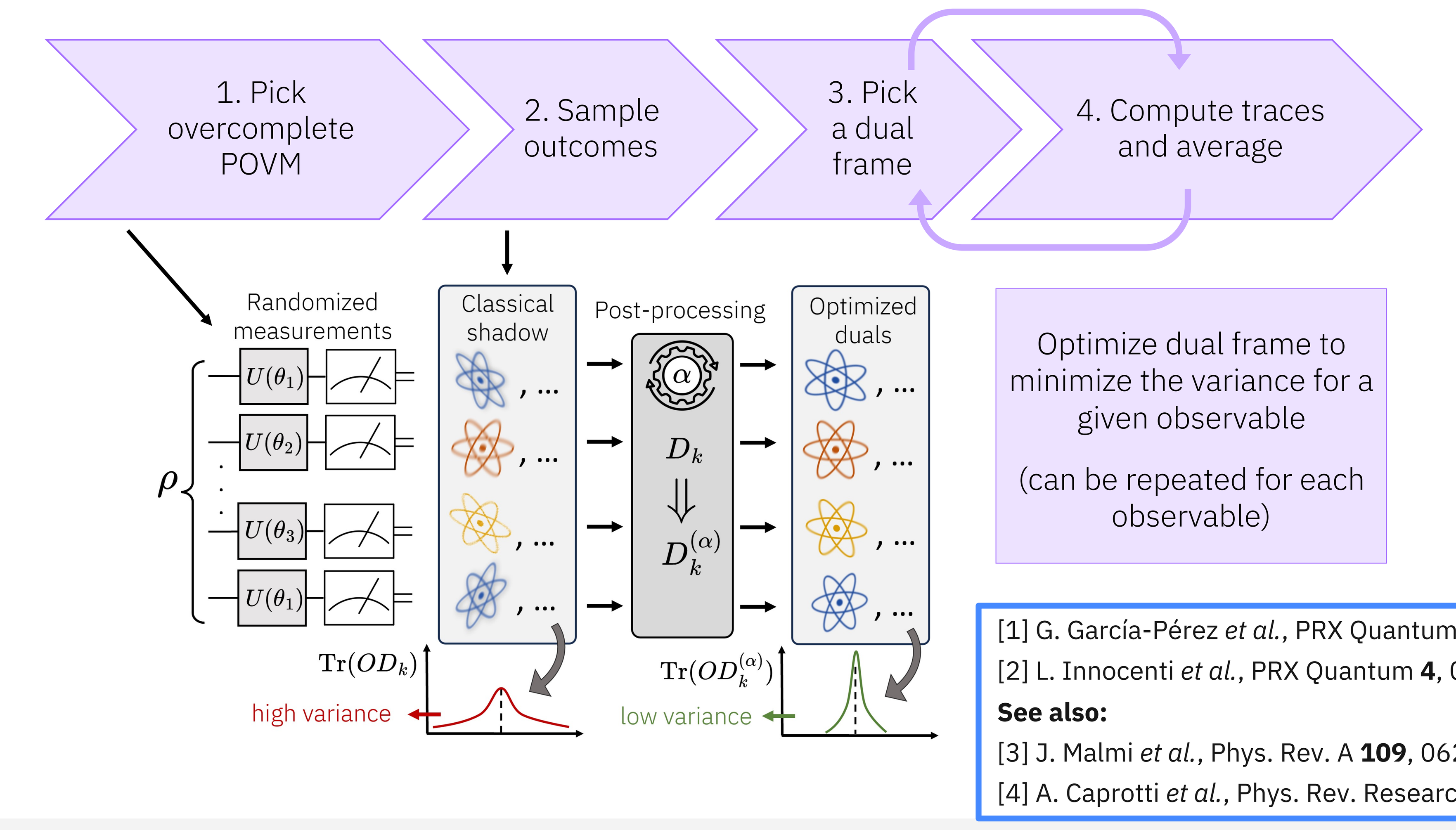
$$\rho = \sum_a \langle \mu_a, \rho \rangle \, \tilde{\mu}_a = \sum_a p_a \tilde{\mu}_a$$

Classical shadows are measurement dual frames!

For an observable $\mathcal{O}$, we can build an estimator via the state estimators, i.e., the POVM duals, as $\bar{o} = \langle \mathcal{O}, \tilde{\mu}_a \rangle$. We then have

$$\mathcal{O} = \sum_k \langle \mathcal{O}, D_k \rangle M_k \quad \longrightarrow \quad \langle \mathcal{O} \rangle = \sum_k \langle \mathcal{O}, D_k \rangle \langle M_k, \rho \rangle = \sum_k \omega_k \mathrm{Tr}[\rho \mathrm{M_k}]$$

The duals give us a systematic way to express observables in terms of POVMs



Optimize dual frame to minimize the variance for a given observable

(can be repeated for each observable)

Parametrizing dual frames

A (non-exhaustive) parametrization of dual frames for a POVM $\{M_k\}$ can be constructed from a probability distribution $\{\alpha_k\}$ as:

$$D_k = \frac{1}{\alpha_k} \mathcal{F}_\alpha^{-1}(M_k) \quad \text{for } k = 1,2,\dots,n$$

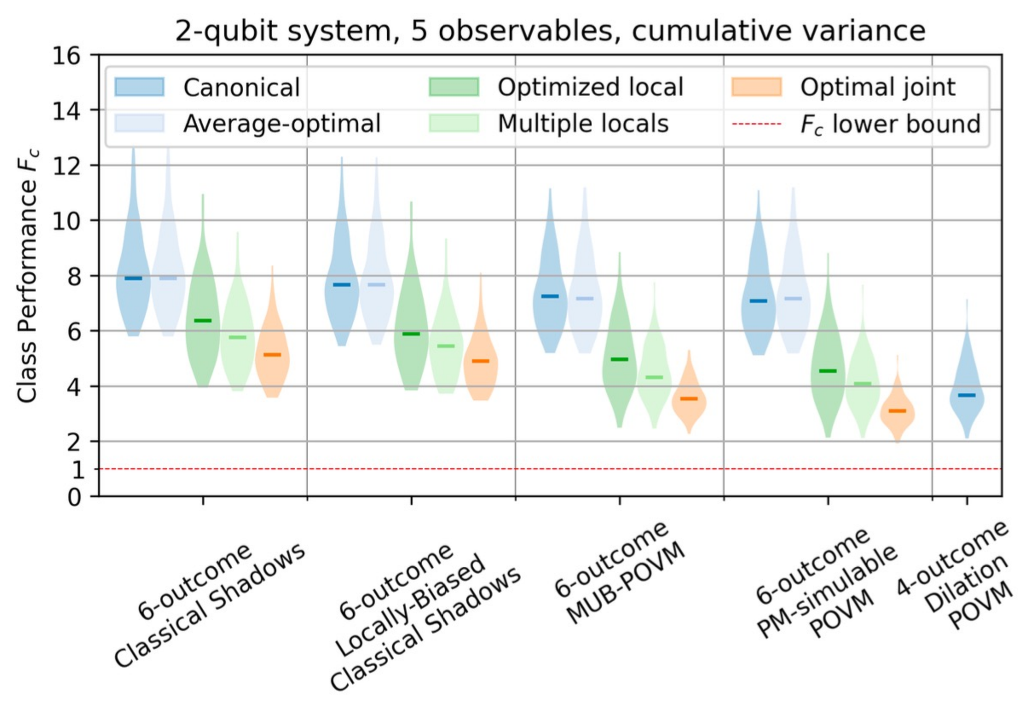
$$\text{where} \quad \mathcal{F}_\alpha : X \mapsto \sum_{k=1}^{n} \frac{Tr[XM_k]}{\alpha_k} M_k$$

- $\alpha_k = 1$: canonical dual frame
- $\alpha_k = \mathrm{Tr}[\rho M_k]$: optimal dual frame given the state ρ [2]
- $\alpha_k = \frac{\mathrm{Tr}[M_k]}{d}$: average optimal dual frame (unknown state)

This is the one that most classical shadows protocol use

Dual frames in action. For random states and observables, optimize $\{\alpha_k\}$ for the best possible single-shot variance

$$F = \underset{\mathbf{M},\mathbf{D}}{\mathrm{argmin}}\ \mathrm{SSV}(\mathbf{M}, \mathbf{D}, \mathcal{O}, \rho) = \underset{\mathbf{M},\mathbf{D}}{\mathrm{argmin}} \sum_k \mathrm{Tr}[\rho M_k]\mathrm{Tr}[\mathcal{O} D_k]^2 - const.$$



- Canonical: $\alpha_k = 1$
- Average-optimal : $\alpha_k = \frac{\mathrm{Tr}[M_k]}{d}$
- Optimized local: α_k optimized as 1-qubit product distribution
- Optimal joint: α_k optimized as global distribution

Common approach for N qubits: use product of 1-qubit POVMs:

$$M_{\mathbf{k}} = M_{k_1 k_2 \cdots k_N} = M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_N}$$

Problem. This does not guarantee a product structure on the duals, which can in general become exponentially expensive to manipulate. Notably, this is the case for the optimal dual frame given ρ. Moreover, Explicit optimization of the dual frame can be cumbersome – especially for adaptive POVMs.

How do we construct efficient and effective duals?

Dual frames from empirical frequencies

Idea: converge to the optimal dual frame $\alpha_k = p_k$ by estimating p_k as frequencies f_k from the actual measurements.

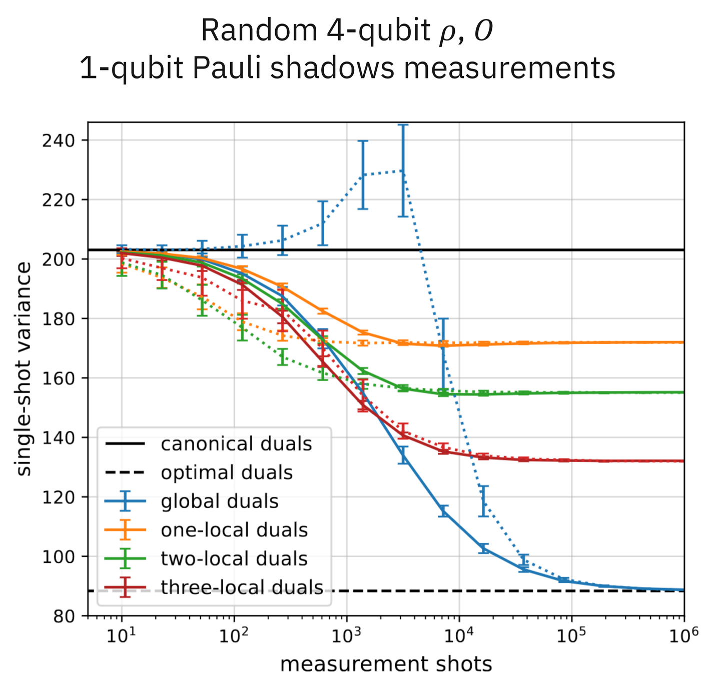
$$|D_k\rangle\rangle = \frac{1}{f_k} \mathcal{F}^{-1} |M_k\rangle\rangle \text{ where } \mathcal{F} = \sum_k \frac{1}{f_k} |M_k\rangle\rangle\langle\langle M_k|$$

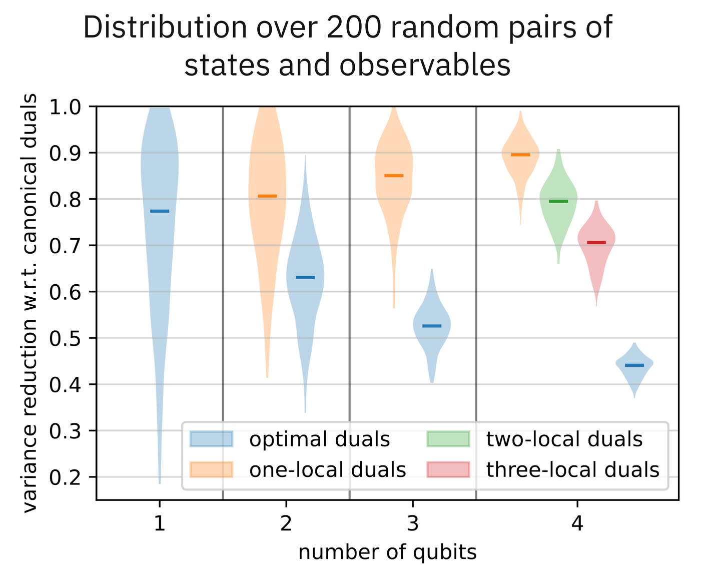
$$f_k\left(\{k^{(1)}, \dots, k^{(S)}\}, S_{\mathrm{bias}}\right) = \frac{\#k + \mathrm{Tr}\left[\frac{1}{d}\mathbb{I} M_k\right]}{S + S_{\mathrm{bias}}}$$

- Restrict correlations in α_k by making it a product of m-local distributions, leading to m-local duals
- Maximize the mutual information of the POVM outcomes between qubits in the same subset.

$$\mathfrak{f}_k^\Lambda = \prod_{i=1}^{l} \mathfrak{f}_{\lambda_i} \qquad \mathfrak{f}_{\lambda_i} = \sum_{\{k_j\}, j \notin \lambda_i} f_{k_1,\dots,k_N}$$

$$I(i,j) = \sum_{k_i,k_j} \mathfrak{f}_{\{i,j\}} \log\left(\frac{\mathfrak{f}_{\{i,j\}}}{\mathfrak{f}_{k_i}^{(i)} \mathfrak{f}_{k_j}^{(j)}}\right) \qquad C(\Lambda) = \sum_{\lambda_i \in \Lambda} \sum_{\substack{j,k \in \lambda_i \\ j \neq k}} I(j,k)$$





[1] G. García-Pérez *et al.*, PRX Quantum **2**, 040342 (2021)
[2] L. Innocenti *et al.*, PRX Quantum **4**, 040328 (2023)
See also:
[3] J. Malmi *et al.*, Phys. Rev. A **109**, 062412 (2024)
[4] A. Caprotti *et al.*, Phys. Rev. Research **6**, 033301 (2024)

Try this at home! Check out our Qiskit POVM toolbox

https://qiskit-community.github.io/povm-toolbox/