Analyzing Ranking Data on Higher-Order Networks Using Quantum Computers

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Introduction—Ranking algorithms are crucial for informed decision-making, as demonstrated by Google's PageRank and Netflix's recommendation system. These systems often operate with noisy or incomplete data, such as pairwise team comparisons in sports rankings. The HodgeRank algorithm [1] overcomes these limitations by utilizing discrete exterior calculus to aggregate incomplete data and derive a global ranking, while also providing a consistency measure. HodgeRank generalizes PageRank and has applications in various domains [2–5].

The mathematical formulation of the HodgeRank problem belongs to the framework of discrete exterior calculus on simplicial complexes [6, 7], which can model complex phenomena in higher-order networks [8–12]. However, higher-dimensional analogues of HodgeRank, defined as k-HodgeRank, computations are exponentially expensive, hindering the exploration of these models.

Here, we propose a quantum algorithm that prepares a quantum state proportional to the k-HodgeRank output, providing a speedup in the dimension of simplices k over classical methods. While the advantage is reduced when extracting classical information, the scaling is maintained for specific tasks. We provide algorithms for measuring rankability, relative ranking, and top-ranking alternatives, with broader applications to higher-dimensional discrete calculus.

HodgeRank Problem—The HodgeRank algorithm objective [1] is to determine a global ranking of a set of alternatives $\mathcal{V} = \{v_1, \ldots, v_n\}$, given a comparison or preference data of the alternatives. The input to the algorithm is the aggregated pairwise preferences of voters, modelled as a data vector defined on the edges, i.e., $\mathbf{s}^1 : \mathcal{E} \to \mathbb{R}$, of a comparison graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The vector component $\mathbf{s}^1([v_i, v_j])$ represents the preference for alternative i over j.

The pairwise ranking problem can be formulated as a weighted ℓ_2 minimization problem

$$\mathbf{s}^1_{\mathrm{G}} = \underset{\mathbf{x}\in \mathrm{im}(\mathbf{B}_1^\dagger)}{\arg\min} \left[\sum_{1\leq i<j\leq n} w_{ij} \left(x_{ij} - \mathbf{s}^1([v_i, v_j]) \right)^2 \right], \tag{1}$$

with the solution $\mathbf{s}^1_{\mathrm{G}}$ given by the projection of $\mathbf{s}^1$ onto the image of the (co)boundary matrix $\mathbf{B}_1^\dagger$. The HodgeRank algorithm then computes the so-called score function $\mathbf{s}^0_*$ by solving the linear system $\mathbf{B}_1^\dagger \mathbf{s}^0_* = \mathbf{s}^1_{\mathrm{G}}$. The global ranking of the alternatives is determined by sorting the component $\mathbf{s}^0_*(v_i)$, with $v_i \succeq v_j$ whenever $\mathbf{s}^0_*(v_i) \geq \mathbf{s}^0_*(v_j)$.

The algorithm can be generalized to higher dimensions, where the comparison graph is replaced by a clique complex $\mathcal{K}_n(\mathcal{G})$, and the data vector $\mathbf{s}^k$ assigns a real number to each k-simplex. The k-HodgeRank solution $\boldsymbol{s}^{k-1}_*$ is given by

$$\mathbf{s}^{k-1}_* = \left(\mathbf{B}_k \mathbf{B}_k^\dagger \right)^+ \mathbf{B}_k \mathbf{s}^k, \tag{2}$$

which assigns a score to each $(k-1)$-simplex in $\mathcal{K}_n(\mathcal{G})$. Computing $\mathbf{s}^k_{\mathrm{G}}$ and $\mathbf{s}^{k-1}_*$ scales as $\Omega(n^k)$ [13, 14], which may hinder the practicality of the algorithm for higher values of k.

Quantum HodgeRank—We build a quantum HodgeRank algorithm on the recently developed quantum algorithm for topological signal processing (QTSP) [15] for analyzing higher-order networks. The algorithm combines the quantum singular value transformation (QSVT) [16, 17] and the projected unitary encoding of the boundary matrices $\mathbf{B}_k$ and $\mathbf{B}_{k+1}$ [18, 19] to prepare, upon successful post-selection in the ancilla qubits, the output state encodes a normalized vector of a vector given in Eq.(2). The cost of this state preparation is given by

$$O\left(n\kappa_k^2 \log(n) \log(\sqrt{n}\kappa_k^2/\varepsilon)\right), \tag{3}$$

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where κ_k is the (effective) condition number of $\mathbf{B}_k$ and $\varepsilon \in (0, 1/2)$ is the error tolerance. To extract information efficiently, we employ this algorithm as subroutines to solve three specific tasks without necessitating the expensive full-state tomography.

Applications—The first application is estimating the consistency measure $\mathrm{R}(k)$ of the ranking data. The value $\mathrm{R}(k)$ quantifies the global consistency of the (sets of) alternatives based on the data, where larger values indicate greater consistency. This measure is derived by projecting the input data onto the subspace corresponding to the HodgeRank solution. Our algorithm applies QTSP and projects the input state to the higher dimensional analogue of Eq. (1). Using the Hadamard test and amplitude estimation, we obtain an ε-approximation to $\mathrm{R}(k)$ with probability of success $1-\delta$. The algorithm requires $O(\kappa_k^2 \log(1/\delta)/\varepsilon^2)$ calls to the state preparation oracle and QTSP algorithm. This provides a polynomial, and potentially exponential, speedup over classical methods for higher k and dense clique complexes.

The next two applications find the L scores of desired alternatives and identify one of the top $\gamma\,\%$ alternatives. We simply employ a Hadamard test and amplitude estimation L times for the first task. This requires $O(L\kappa_k^2/\sqrt{n}, \log(1/\delta)/\varepsilon')$ calls to the state preparation oracle, quantum HodgeRank, and single-qubit gates. For the second task, we find the threshold for the top $\gamma\,\%$ alternatives and create a uniform superposition state of all such alternatives. Measuring this state in the computational basis gives the desired alternative. The total cost of outputting this state with a high probability of success and precision parameter Δ is $\tilde{O}\big(\kappa_k^2/(\gamma\sqrt{n}\Delta)\big)$ calls to two quantum state preparation oracles and the quantum HodgeRank subroutine (with the cost shown in Eq. (3)).

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