Extended abstract: Quantum Policy Gradient in Reproducing Kernel Hilbert Space

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Reinforcement learning (RL) is a technique that is successful across a wide range of interactive applications. A key limitation of RL is that it requires a large number of samples before a high-performing policy is learned. With the aim of reducing the sample complexity, several works have proposed applying RL systems within quantum-accessible environments, where interactions with the environment occur within a quantum system allowing to make use of superpositions across state-action trajectories. While exponential sample complexity improvements have only been shown for a special case environment formulated around Simon's problem [1], recent policy gradient algorithms demonstrate benefits in terms of quadratic sample complexity improvements when applying a parametrised quantum circuit within a quantum environment due to the properties of quantum superpositions [2]. Moreover, several quantum RL works demonstrate that by using parametrised quantum circuits, the number of parameters can be reduced compared to using classical neural networks [3, 4] – although this line of work focused primarily on classical environments.

Despite the promise of quadratic or better improvements, limited work has been done in quantum-accessible environments, and especially in the construction of suitable parametrised quantum circuits (PQCs). Previous work has introduced various PQCs for classical RL [5], using hardware efficient PQCs with alternating layered architecture. Two of these circuit classes, namely Raw-PQC and Softmax-PQC, have then been analysed further in the context of quantum-accessible environments when using quantum policy gradient (QPG) algorithms [2], which yield quadratic improvements in query complexity, i.e. the calls to the oracle to estimate the policy gradient.

Due to quantum states residing in a high-dimensional complex Hilbert space, parametrised quantum circuits have a natural interpretation in terms of kernel methods. While so far, this property has been discussed widely for supervised learning [6, 7], this has not yet been adopted in reinforcement learning.

Our work is inspired by streams of work in classical RL that use kernel-based formulations of the policy [8, 9]. We formulate Gaussian and softmax policies based on quantum kernels and analyse their efficiency across various optimisation schemes with quantum policy gradient algorithms. While maintaining quadratic speedups associated with QPG, the use of quantum kernels for the policy definition leads to advantages such as analytically available policy gradients, tunable expressiveness, and techniques for sparse non-parametric representations of the policy within the context of vector-valued state and action spaces. This also leads to a quantum actor-critic algorithm with an interpretation as performing a natural gradient. Unlike quantum algorithms for natural policy gradient [10, 11], the proposed algorithm is formulated within the kernel method framework and is tailored to the quantum accessible environment where it can exploit a quadratic sample complexity improvement as well as a variance reduction as is often associated with actor-critic RL.

Using quantum kernels for reinforcement learning policies.– Kernel methods have strong theoretical foundations for functional analysis and supervised learning (see e.g. [12] for an overview). We review some of these useful properties here and how they can be applied to formulate and learn efficient policies for quantum reinforcement learning.

First, the kernel function determines the expressible function space through its reproducing kernel Hilbert space (RKHS). The choice of the kernel function thereby provides an opportunity to balance the expressiveness, training efficiency, and generalisation. For instance, reducing the bandwidth factor to $c < 1$ of the squared cosine kernel

$$\kappa(s, s') = \prod_{j=1}^{d} \cos^2(c(s - s')/2) \tag{1}$$

restricts features to parts of the Bloch sphere, allowing improved generalisation [13]. As this affects the coverage of the Bloch sphere, this tuning factor can be related to expressiveness measures based on the Haar distribution, such as the frame potential or the KL difference [14]. Tuning a single parameter is significantly more convenient compared to redesigning the ansatz of a PQC.

Second, kernel functions inherently define a particular feature-map. This interpretation follows from Mercer's theorem, which states that every kernel function can be written as

$$\kappa(s, s') = \sum_{i=1}^{\infty} \lambda_i e_i(s) e_i(s') \,,$$

where for all i, e_i is an eigenfunction such that $\lambda_i e_i(s') = \mathscr{T}_K[e_i](s') = \int_a^b \kappa(s, s') e_i(s) ds$ with eigenvalue λ_i. Mercer's

theorem leads to the kernel trick,

$$\kappa(s, s') = \langle \phi(s), \phi(s') \rangle \,,$$

which allows writing the kernel function as an inner product based on a feature-map ϕ. For quantum kernels, this allows a definition of kernels in terms of the data encoding as a feature-map. For instance, the basis encoding corresponds to the Kronecker delta kernel; the amplitude encoding corresponds to the inner product quantum kernel; the repeated amplitude encoding corresponds to a polynomial kernel; the coherent state encoding corresponds to a Gaussian Kernel; and the product encoding corresponds to a cosine kernel [7].

Third, the kernel based formalism is based on a representative set of state-action pairs. In a supervised learning setting, the representer theorem guarantees that the optimal function approximator can be written as linear combination of kernel evaluations based on input data pairs, which leads to the formulations of support vector machines and kernel regression. In quantum supervised learning, one uses this property to evaluate the kernel in a quantum device and the model is then computed from the kernel in a quantum or a classical device [6, 15]. In a quantum reinforcement learning setting, we analogously consider the optimal deterministic policy μ as a linear combination of kernel computations with regard to a select subset of the states:

$$\mu(s) = \sum_{i=1}^{N} \beta_i \kappa(s, s_i) \,.$$

Using this quantity as the mean of a Gaussian distribution allows the quantum analogue of the Gaussian policies that are popular in classical reinforcement learning with vector-valued action spaces. Moreover, it comes with analytical forms for the gradient and is suitable for various non-parametric optimisation schemes.

Fourth, with the aim of performing ℓ_2-regularisation as a subroutine for sparsification within a non-parametric learning algorithm, we make use of the result that for every RKHS $\mathscr{H}_K$ with reproducing kernel K, and any $g \in \mathscr{H}_K$,

$$||g||^2_{\mathscr{H}_K} = \langle g, g \rangle_{\mathscr{H}_K} = \int (\mathscr{R}g(x))^2 dx$$

where the operator $\mathscr{R} : \mathscr{H}_K \to \mathscr{D}$ can be interpreted as extracting information from the function value which gets penalised during optimisation [12]. For instance, it can penalise large higher or lower order derivatives, large function values, or still other properties, leading to smoother optimisation landscapes and therefore improved convergence to the global optimum. This property can be exploited when directly optimising the kernel.

Overview of the contributions.– This work contributes the following theoretical and empirical results to the field of quantum reinforcement learning:

- We propose two classes of kernel-based policies. First, we propose Representer PQCs, which incorporate representer theorem based formalisms directly within a quantum circuit and which are suitable for both analytical and numerical gradient based optimisation. Second, we propose Gaussian kernel-based policies based on a classically known mean function and covariance, which has known analytical gradient removing the need for expensive estimation procedures required for analytical quantum policy gradient with traditional PQCs. We also provide a formula to scale the number of representers based on a Lipschitz constant.

- We use a central differencing approach on phase oracles of the value function for a numerical quantum policy gradient algorithm [2] based on Representer PQCs. We report a query complexity comparable to Jerbi et al. [2] but note the potentially lower number of parameters.

- We use analytical quantum policy gradient algorithms which perform quantum multivariate Monte Carlo schemes on binary oracles of the policy gradient in quantum-accessible environments. We first confirm that applying quantum analytical policy gradient to kernel-based policies yields a query complexity that is comparable to Jerbi et al. [2].

- We then proposes two further improvements. For reducing the parameter dimensionality, we propose a vector-valued kernel matching pursuit. For reducing the variance of the policy gradient due to the variability in the cumulative reward, we propose Compatible Quantum RKHS Actor-Critic, an approach which performs analytical quantum policy gradient with an oracle that performs occupancy-based sampling in a quantum device and includes the critic's prediction rather than the cumulative reward of the trajectory. Theorem 6.1**a** (see main text) demonstrates that the resulting query complexity depends the maximal deviation from a baseline estimate, rather than on the maximal cumulative reward. Theorem 6.1**b** (see main text) provides an improved result which exploits an upper bound on the variance of the gradient of the log-policy, and thereby demonstrates how designing smooth policies such as the Gaussian kernel-based policy can give additional query complexity benefits.

TABLE I: Query complexity of policy gradient estimation with PQCs. General notations: $\mathscr{A}$ is the action space; $r_{\max}$ denotes the maximal reward; T is the horizon; d is the parameter dimensionality; γ is the discount factor; and ϵ is the tolerance for error in the gradient estimate. Specific notations: $\mathscr{T}$ is the temperature of the softmax; D is an upper bound on higher-order derivatives of the policy; ϵ_Q is the maximal absolute deviation of the critic's prediction to a baseline estimate; B_p is a p-norm upper bound on the gradient of the log-policy; N is the number of representers in a kernel policy; σ_{∇_p} is an upper bound on the p-norm on the standard deviations of the partial derivative of the log-policy; C_p is a p-norm upper bound on the gradient of the critic; and $\xi(p) = \max\{0, 1/2 - 1/p\}$ is used for converting across p-norms.

Algorithm	**Oracle and estimation**	**Query complexity**
1. Policy gradient with Softmax-PQC [17]	Return oracle, single-qubit parameter shift rule [18], and classical Monte Carlo	$\tilde{\mathcal{O}}\left(\frac{\mathscr{T}^2 r_{\max}^2 T^2}{\epsilon^2(1-\gamma)^2}\right)$
2. Numerical QPG and Raw-PQC [2]	Return oracle, quantum gradient estimation via central differencing [19]	$\tilde{\mathcal{O}}\left(\sqrt{d}\frac{DTr_{\max}}{\epsilon(1-\gamma)}\right)$
3. Analytical QPG and Softmax-PQC [2]	Analytical gradient oracle, bounded quantum multivariate Monte Carlo (Theorem 3.3 [20])	$\tilde{\mathcal{O}}\left(d^{\xi(p)}\frac{B_p T r_{\max}}{\epsilon(1-\gamma)}\right)$
Proposed: Numerical QPG in RKHS	cf.2	cf.2 but $d = NA$
Proposed: Analytical QPG in RKHS	cf.3	cf.3 but $d = NA$
Proposed: Compatible Quantum RKHS Actor-Critic	Analytical gradient oracle, near-optimal quantum multivariate Monte Carlo (Theorem 3.4 [20])	$\tilde{\mathcal{O}}\left(d^{\xi(p)}\frac{\epsilon_Q B_p}{(1-\gamma)\epsilon}\right)$ for $d = NA$
		$\tilde{\mathcal{O}}\left(\frac{d^{\xi(p)}\epsilon_Q \sigma_{\nabla_p}}{(1-\gamma)\epsilon}\right)$ for $d = NA$
Proposed: Deterministic Compatible Quantum RKHS Actor-Critic	cf.3	$\tilde{\mathcal{O}}\left(d^{\xi(p)}\frac{C_p}{(1-\gamma)\epsilon}\right)$ for $d = NA$

- We further propose Deterministic Compatible Quantum RKHS Actor-Critic, which is based on the deterministic policy gradient theorem [16]. Theorem 6.3 (see main text) demonstrates that the resulting query complexity depends on the number of representers and the maximal gradient norm of the critic, which illustrates the importance of techniques such as kernel matching pursuit and regularisation.

Our query complexity results compare favourably to other policy gradients methods for PQCs (see Table I).

Why QTML.– This paper presents numeric and analytical optimisation techniques for quantum kernel policies for efficient reinforcement learning in vector-valued action spaces. We prove quadratic improvements of kernel-based policy gradient and actor-critic algorithms over their classical counterparts, across different formulations of stochastic and deterministic kernel-based policies. Two actor-critic algorithms are proposed that improve on quantum policy gradient algorithms under favourable conditions. The proposed quantum kernel policies allow convenient analytical forms for the gradient and techniques for expressiveness control. Considering the significance for the quantum machine learning community, we believe our work is well-suited for QTML.

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